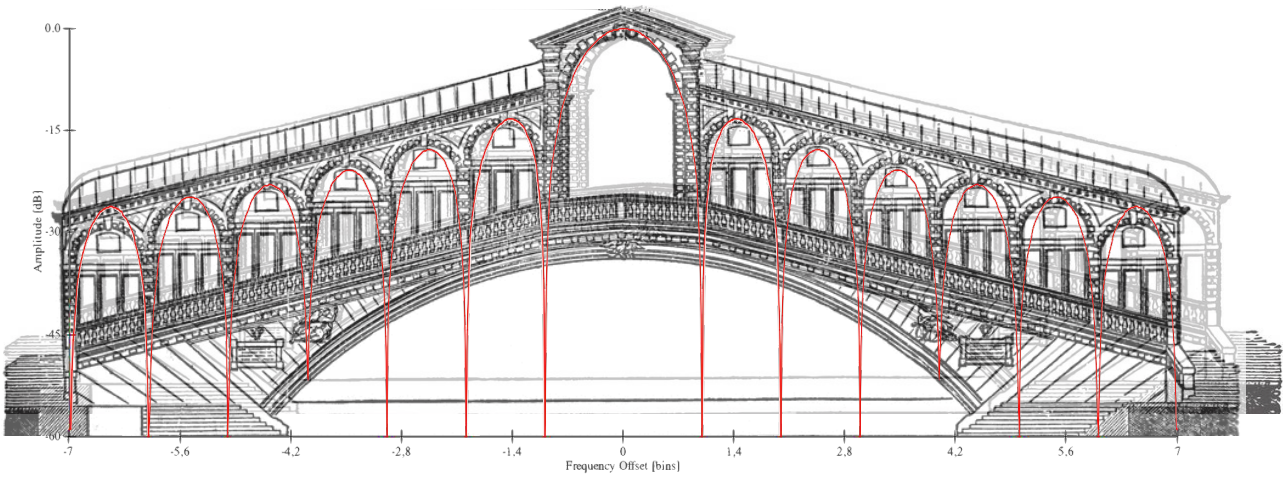
Fourier Round Trip Program

Alan Shepherd
https://ashepherd.eu

Like a circle in a spiral, like a wheel within a wheel
Never ending or beginning on an ever spinning reel
As the images unwind, like the circles that you find
In the windmills of your mind!

From the lyrics of the song "The Windmills of your Mind"
by Alan Bergman, Marilyn Bergman
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Frontispiece 1 Rectangular Window Transform Superimposed on Rialto Bridge, Venice, Italy
Bridge adapted from ClipArt ETC FloridaCenter for Instructional Technology, Free Classroom License
https://etc.usf.edu/clipart/26200/26244/rialto_26244.htm

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1 Introduction

The Fourier Round Trip program is intended for exploring wave forms produced by harmonics or overtones, for example the sound waves produced by musical instruments, or any other type of wave.

It has two main functions with the following features:

- Synthesis to build up a wave from its harmonics:
  - Visualisation of the component waves, frequency spectrum, resultant wave
  - Visualisation of the constructor and generator figures.
  - Traverse the angles showing the vectors of the reference figure and the projection of the constructor figure to the wave.
  - Preset the component waves for sawtooth (all harmonics), square (odd harmonics) or triangular waveforms. Sawtooth and square waves can have linear or exponential amplitude progressions.
  - Add random noise.
  - Edit the frequency, amplitude and phase of each harmonic.
  - Sort the harmonics by frequency, amplitude or phase.
  - Export the harmonics to a CSV-file and import them.
  - Play the resultant wave at any frequency and any duration.
  - Add an exponential decay to the sound.
  - Save the resultant wave to a WAV sound file.
  - Send the resultant wave directly to the Analysis function.
- Analysis to perform Fourier analysis of a sound file or wave from Synthesis:
  - Read a WAV sound file and display the waveform of left or right channel.
  - Display the meta-data from the file.
  - Zoom and pan the wave plot over time. (x-axis)
  - Zoom the wave plot amplitude (y-axis).
  - Select a window (or frame) for analysis.
  - Apply a variety of window-functions to the frame.
  - Show the window-function, its transform and main parameters.
  - Pad the frame with zeros to increase frequency resolution.
  - Play any section of the sound once or as a loop.
  - Perform a Discrete Fourier Transform (DFT).
  - Perform a Fast Fourier Transform (FFT).
  - Perform a Chirp Z-Transform (CZT) of part of the frequency spectrum.
  - Perform a Short Time Fourier Transform STFT over multiple frames of DFT, FFT or CZT.
  - Average the transform of multiple frames of DFT, FFT or CZT.
  - Apply a DC filter before transform.
  - Visualise analysis results as frequency and phase spectra.
  - Visualise the STFT result in an animated spectrum.
  - Zoom and pan the frequency axis of the resulting spectrum.
  - Use linear or logarithmic (dB) scales for amplitudes.
  - Find peaks in spectrum and interpolate them with a choice of methods.
  - Show inharmonicity values for interpolated peaks as a coefficient and in cents.
  - Show the cumulative line spectrum.
  - Export the analysis results as a CSV-file.
  - Export and import all analysis settings so that analysis results are reproducible.
  - Send analysis results to the Synthesis function.
  - Superimpose the harmonics of any fundamental frequency onto the spectrum.
  - Superimpose notes of the scale in various temperaments onto the spectrum.

- Analysis of Decay Rate of a sound file
    - Averaging of the wave with variable span.
    - Linear interpolation of decay.
    - Linear or Logarithmic (decibel) scales.

In combination, the synthesis and Fourier analysis enable a "round trip" of synthesising a wave, sending it to the analysis, sending the harmonics back to the synthesis.

What it is not

We follow the philosophy that a program should perform one or more specific functions and not try to perform other, possibly related, functions that other programs already do well. Instead, it should be open to exchanging data with these other programs. Therefore, this program:

- Is not a synthesiser with attack, decay, filters etc., but the generated waves could probably be used as samples for such programs.
- Does not have a MIDI interface to play music, but again, the results could be used as samples for such programs.
- Does not edit sound files but reads and writes WAV-files which can be edited and converted with other programs.
- Does not perform real-time Fourier analysis from a microphone, there are other programs for that, and we are interested in a single cycle or a few cycles that we wish to select from a recorded or synthesised wave. Recordings made with other programs and saved as a WAV file can, of course, be used.

In keeping with this philosophy, the synthesised harmonics, and the analysed harmonics can be exported to CSV files for further analysis, e.g. in Excel, and the synthesised wave can be output to a WAV file.

Some important terms are defined in chapter 2. Chapter 3 gives the theoretical background of the program and chapter 4 describes some possible uses. The installation of the program is described in chapter 5. An overview of the main window and menus is given in chapter 6. Chapter 7 describes the synthesis function and chapter 8 the analysis. Chapter 10 introduces some experiments to demonstrate the purpose of some of the functions. The appendices give further mathematical background and a list of the windowing functions provided.

2 Definitions

Frame	1) In a sound file, the bytes that make up a sample. 2) In analysis, a section of the wave being analysed, equivalent to a Window (see below). The term “frame” is used when the window is shifted in time for repeated analysis.
Fundamental	The natural frequency with which a string or pipe etc. vibrates.
Harmonic	A multiple N of the fundamental frequency. For N=1 the first harmonic is the fundamental.
Overtone	A multiple M of the fundamental frequency. For M=1 the first overtone is twice the fundamental frequency and is the second harmonic.
Partial	Synonym for Overtone.
Spectrum	The component sine (or cosine) waves that make up a wave, each given by frequency, amplitude and phase.
Bin	A division of the spectrum frequencies. The bin contains the amplitude contributions of the frequencies that fall within that bin. Its width (on the frequency axis) is determined by the resolution.
Reference Figure	The figure traced out by the end of the vector resulting from the addition of the vectors of the component waves.
Resultant	The wave resulting from the addition of its component harmonics.
Constructor Figure	The figure that can be used to construct a waveform by laying off the angles and plotting the vertical displacement.
Generator	An electronic device or computer program that produces a certain waveform, e.g., sine wave, square wave. Also known as Function Generator.
Window	A short section of sound, e.g., selected from a music file and possibly modified with a window-function. (The word is also used for a part of a computer display.)
Window-function	A function used to smooth the ends of a window to avoid discontinuities when it is putatively extended in time.

3 Theoretical Background

3.1 Introduction

Musical notes are sound waves of certain frequencies caused by vibrating strings or tubes of air. The timbres of musical instruments are caused by the presence of harmonics or overtones in addition to the basic frequency. Apart from acoustics, there are many other areas where waves occur.

The simplest waveform is the sine wave, and all waveforms can be decomposed into a set of sinewaves at different frequencies and amplitudes by Fourier analysis. The result is a frequency spectrum.

A sinewave can be constructed from a circle: the x-axis is the angle going anti-clockwise from 0° to 360° or from 0 to 2π radians. The y-axis is the sine of the angle. The circle is traversed once for each cycle of the wave. A rotation rate of 2π radians per second equals a frequency of the wave of 1Hz (Hertz or cycles per second). The circle is a representation of a sine wave in polar coordinates, with the angle θ and the length *r*.

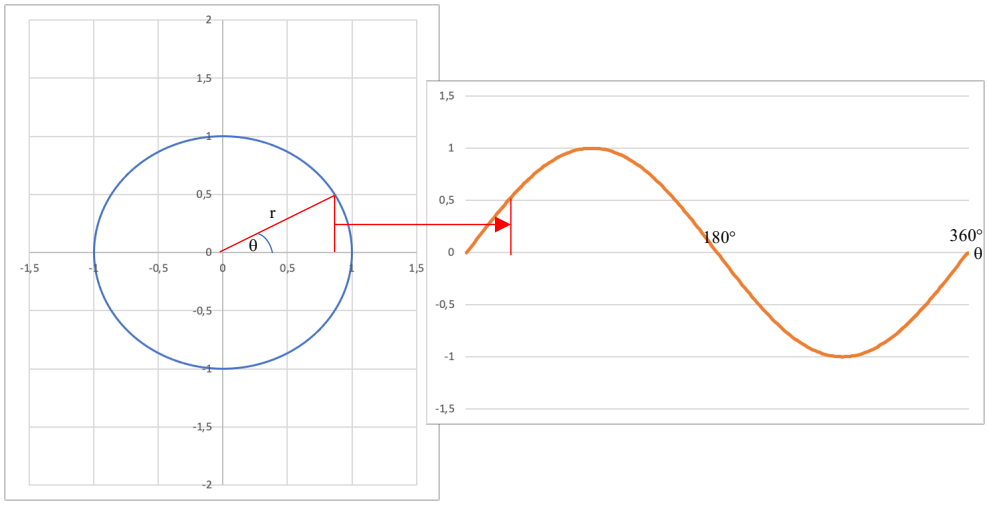


Figure 3-1Circle and Derived Sine Wave

The circle is called the **Reference Figure** of the sine wave.e

3.2 Occurrences of Waves

The sine wave also results from harmonic oscillation such as a mass on the end of a spiral spring, a pendulum, or some electronic circuits. They occur in large bodies of water such as lakes and oceans. Electromagnetic waves range through radio waves, microwaves and heat, light, ultra-violet, x-ray and gamma-ray radiation. They occur as gravitational waves in cosmology and seismic waves resulting from earthquakes.

Wave functions are used in physics to mathematically describe the behaviour of sub-atomic particles and quantum effects.

3.3 Harmonics

A string, for example on a guitar, is fixed at both ends and so can only vibrate with its full amplitude in the middle. The modes of vibration are therefore restricted as shown in Figure 3-2. The string can vibrate at 1, 2, 3, … times the fundamental frequency (we see ½, 1, 1½, … wavelengths).

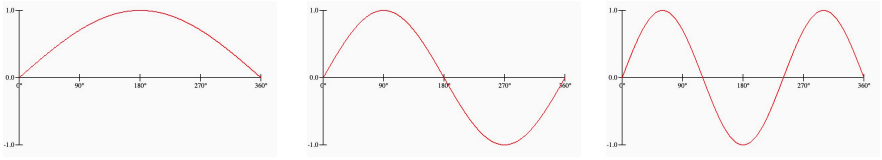


Figure 3-2 First Three Modes of Vibration of a String

The air in a pipe, as in a church organ, closed at one end and open at the other, can only vibrate as shown in Figure 3-3, where the pipe is horizontal with the open end at the right. The y-axis represents the amplitude of the compressions of the air at the point along the x axis. This can vibrate at 1, 3, 5, … times the fundamental frequency (we see ¼, ¾, 1¼, … wavelengths).

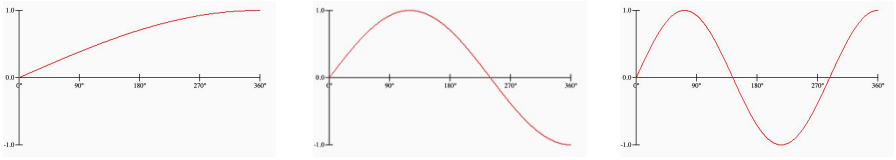


Figure 3-3 Vibration Modes of a Pipe Open at One End

3.4 Waveforms

Two common waveforms are the sawtooth and the square wave. The sawtooth sounds like a violin and has all the harmonics, and the square wave is the basic sound used in electronic organs and has all the odd harmonics.

To generate a sawtooth wave the octave harmonics are added with successively reduced amplitudes:

$$y = \sin(\theta) + \frac{1}{2}\sin(2\theta) + \frac{1}{3}\sin(3\theta) + \cdots$$

The reference figures of the harmonics are a circle of half the radius being traversed at twice the frequency, a circle of one third of the radius being traversed at three times the frequency, etc.

For a square wave, only the odd harmonics are used:

$$y = \sin(\theta) + \frac{1}{3}\sin(3\theta) + \frac{1}{5}\sin(5\theta) + \cdots$$

It is used in electronic organs because a square wave is easy to generate electronically by switching a circuit on and off.

The triangular wave has odd harmonics of alternating signs with amplitudes $\frac{1}{N^2}$:

$$y = \sin(\theta) - \frac{1}{3^2}\sin(3\theta) + \frac{1}{5^2}\sin(5\theta) - \cdots$$

Another variant is an approximation to a harpsichord sound given in [Sethares p.233[1] as

$$y = \sin(\theta) + a^2\sin(2\theta) + a^3\sin(3\theta) + \cdots$$

With the amplitude factor for harmonic n being $a^{(n-1)}$ where a is between 0.75 and 0.9.

[1] "The amplitude of the partials is assumed to die away at a rate of $.75^n$, where n is the partial number. Surviving historical harpsichords vary considerably in these parameters. The low strings of some have more than 80 discernible partials, decreasing with an exponent as high as 0.9, whereas the strings of others display as few as 8 partials with a more rapid decay."

[Benade] p.100 ff. also gives the relative amplitudes for plucked and struck strings depending on the plucking or hitting point. For struck strings the harmonics have amplitudes proportional to $1/n^2$ but the harmonics actually present depend on the plucking or striking point along the string. On p. 103 [Benade] gives a progression of 1/n for a string struck by a hard hammer. Thus more higher harmonics are produced by striking the string than by plucking it.

3.5 Questions

My original question when I started this research was:

- If a sine wave is constructed from a circle, from what figures are other waves constructed?

And arising from this:

- Can they tell us anything useful?
- Can they be related to other waves such as earthquakes, quantum wave equations, etc.?
- In quantum physics, where particles have a wave representation, what does the wave represent? What does the corresponding polar figure represent?

3.6 Vector Representation

The reference figure can be represented in polar coordinates. Adding harmonics is equivalent to adding the vectors of the component sine waves.
Figure 3-4 below on the left shows the vectors to the reference circles for the angle of 10° for the fundamental $\sin(10°)$ in blue, the second harmonic $\frac{1}{2}\sin(20°)$ in red, the third harmonic $\frac{1}{3}\sin(30°)$ in purple and the fourth harmonic $\frac{1}{4}\sin(40°)$ in yellow. The addition of the vectors is shown on the right.

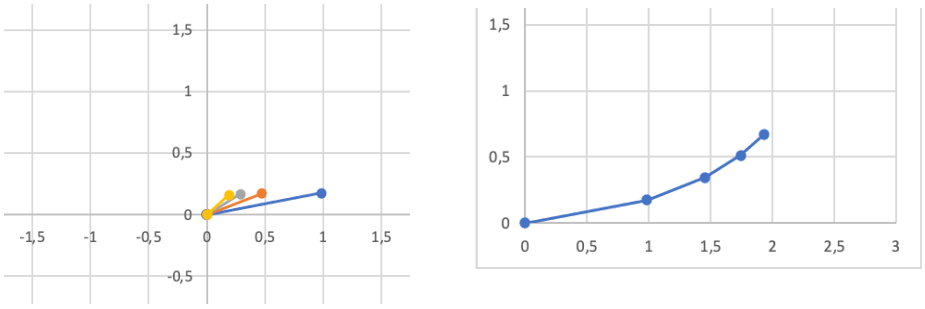


Figure 3-4 Vectors of 4 Harmonics

Note that the sum of the vectors does not arrive at the figure at the point indicated by the fundamental angle (10°), but at a point further round (see Figure 3-5).

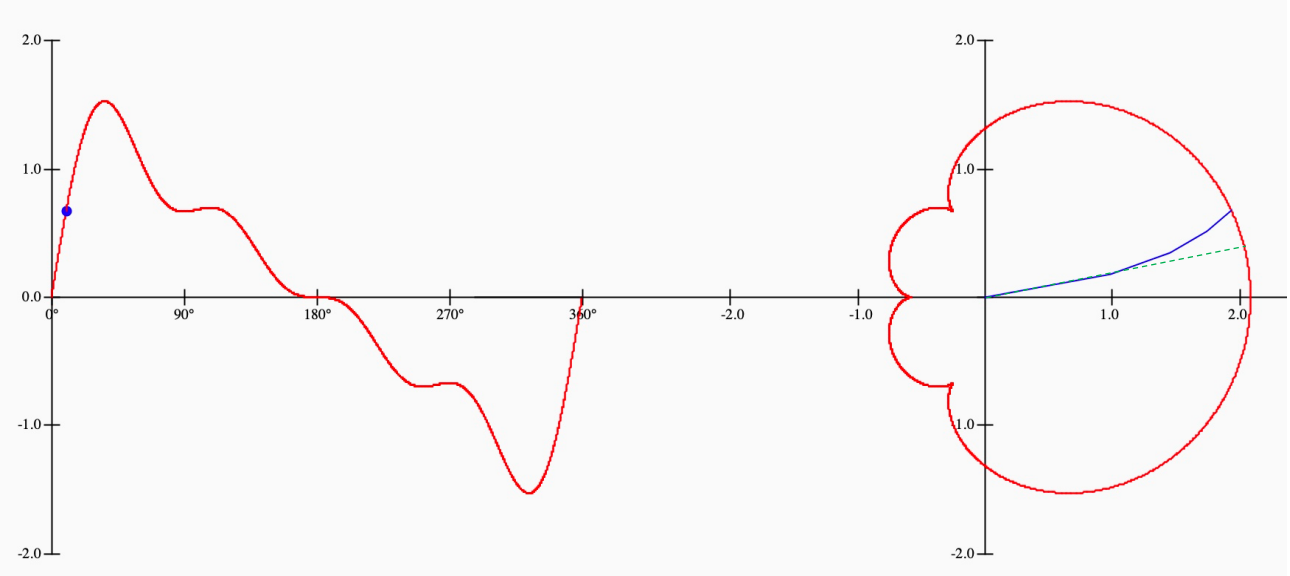


Figure 3-5 Vector of 4 Harmonics with Resultant Wave and Reference Figure

3.7 Constructor Figure

To obtain the constructor figure, i.e., the figure from which one can derive the resultant wave in the same way as the sine wave is derived from the circle, we must produce (extend) the first vector, sin(x), until it has the same y-value as the resultant vector.

For a wave function of the form $f_s = \sin(\theta) + \frac{1}{2}\sin(2\theta) + \frac{1}{3}\sin(3\theta) + \cdots$ or similar,

$$y = f_s(\theta)$$

$$r = \frac{y}{\sin(\theta)} = \frac{f_s(\theta)}{\sin(\theta)}$$

$$x = \sqrt{r^2 - y^2}$$

When using this, the sign of x must be changed to negative for $90° < \theta < 270°$.

Alternatively

$$x = \frac{y}{\tan(\theta)}$$

In which x is not defined for $\theta = 0°$ and $\theta = 180°$, where $\tan(\theta) = 0$. There can also be precision problems when $\tan(\theta)$ becomes very small or very large.

In the Figure 3-6 the red shape is the result from the vectors with the point for 30° shown at the end of the blue vectors. The orange shape is the constructor figure which could be used to construct the wave by laying off the angles – the orange dot is the point for 30° and has the same y-value as the end of the blue vectors. It can be seen that the two figures are different!

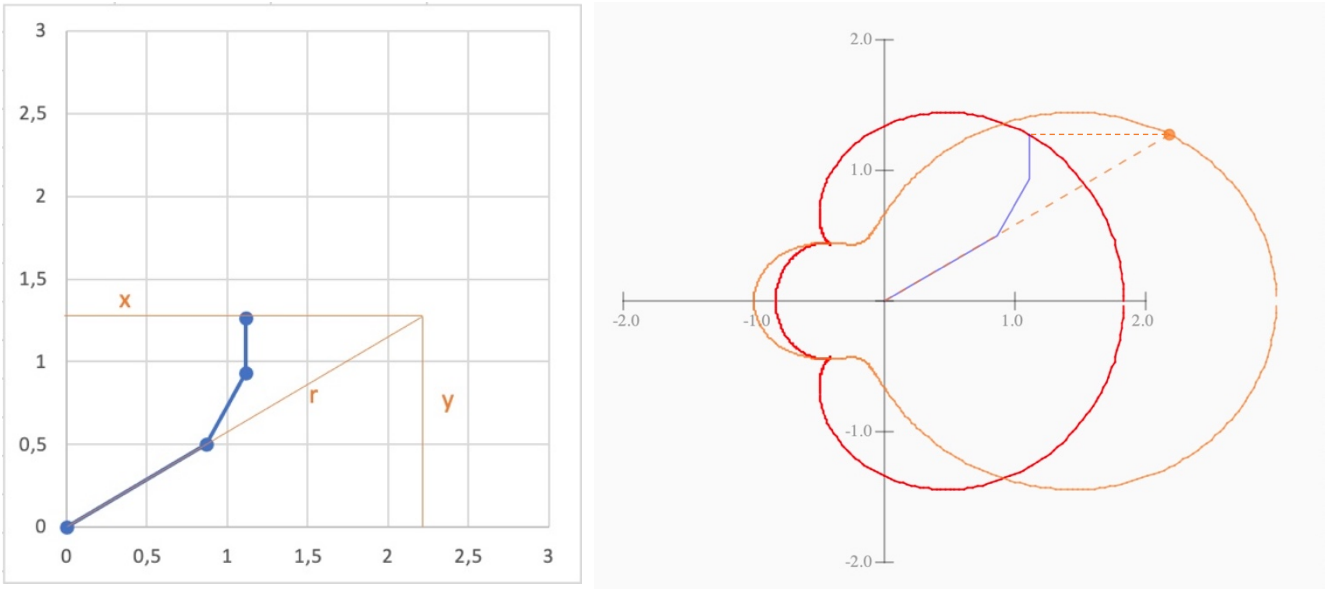


Figure 3-6 Derivation of Constructor Figure for 3 Harmonics

3.8 Incomplete Constructor Figures

It is not always possible to obtain a complete constructor figure for a given wave. If the amplitude of the wave is negative at any angle between 0° and 180°, or if the amplitude is positive at any angle between 180° and 360°, the line at the given angle will never reach the y-value in the Cartesian two-dimensional space. Figure 3-7 shows this for a wave synthesised from frequences of 2 and 3 of equal amplitude the line of the constructor figure at 110° must be followed upwards until it reaches the height of the wave, but the wave is negative at that point so can never be reached. The program shows this as a dotted line at the angle and the text “No valid constructor”.
Also, as the y-value approaches zero, the length of the line becomes infinite or undefined (hence the gap at the right in Figure 3-6[2] and Figure 3-7).

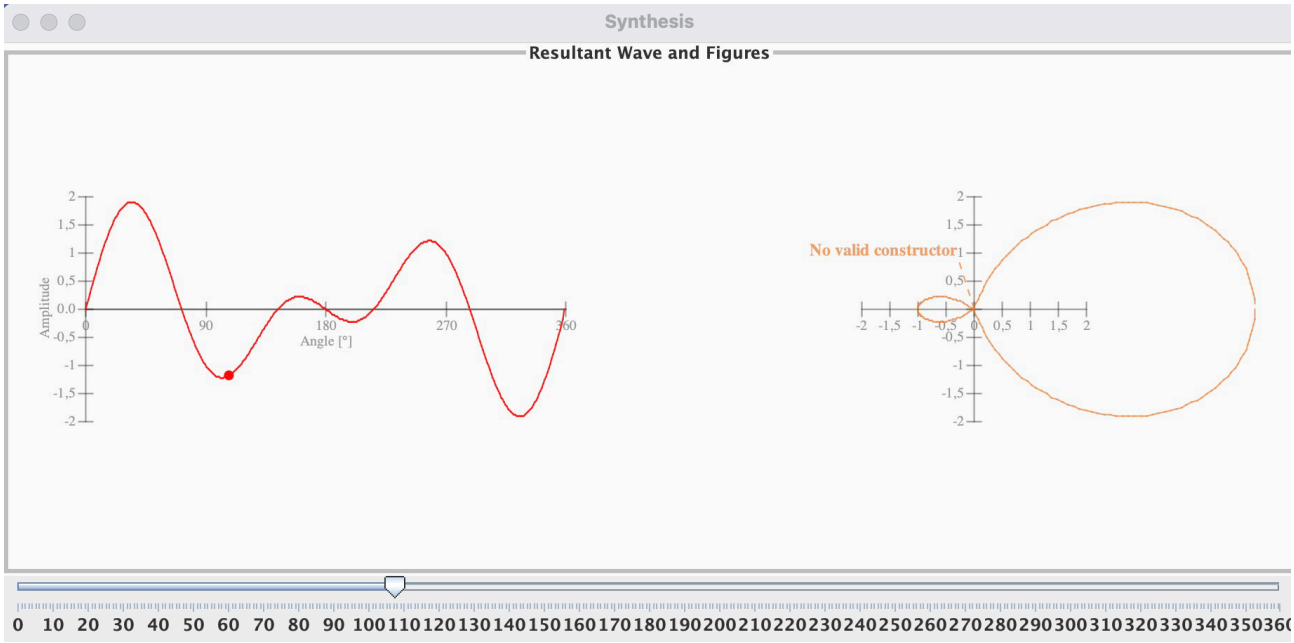


Figure 3-7 Wave with no Valid Constructor

[2] There is no gap on the left at 180° because the line has been drawn from the previous valid point to the next valid point. The occurrence at 0° was not treated in this way because this is the starting point.

Question: Is there a different coordinate space that would make this work, e.g. a cylinder or sphere or torus?

3.9 DC in Constructor Figures

A special case of this problem occurs when a wave that would have a constructor figure contains DC (direct current) components, i.e., values that do not change with time and therefore have a frequency of 0. To deal with this, we can initially omit the DC components from the calculation of the constructor figure and add the total DC at the end to shift the whole figure up or down the y-axis accordingly. This also shifts the origin of the constructor figure. Figure 3-8 shows a wave with a DC component of 1.0 and two harmonics:

$$y = 1.0 + \sin(x) + \tfrac{1}{2}\sin(2x).$$

For the red reference figure, the first vertical vector from the origin up the y-axis to 1.0 is the DC component and the two harmonics go from there. For the orange constructor figure, the whole figure is shifted up 1.0 by the DC component. (Compare this to Figure 3-11.)

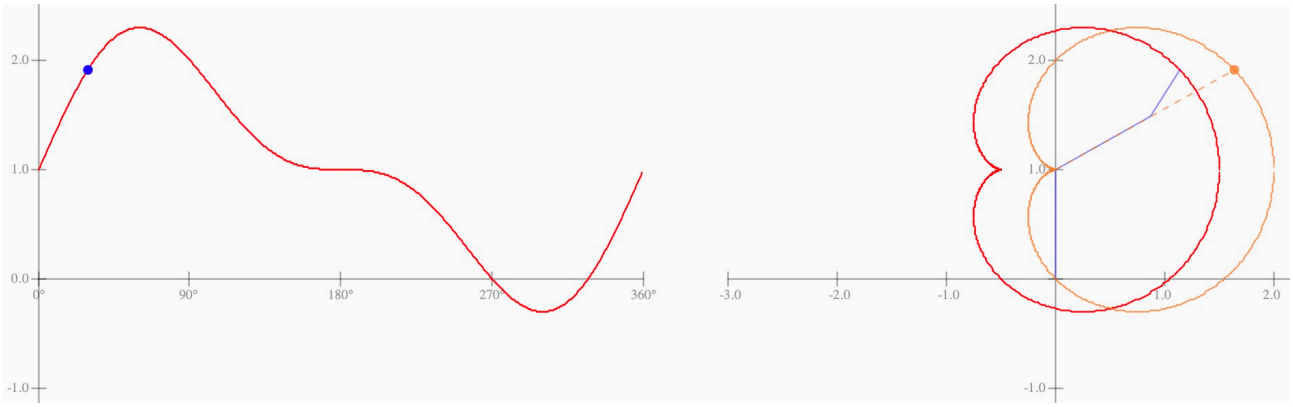


Figure 3-8 A DC Component of 1.0 and Two Harmonics

For angles where there is no valid constructor, this is indicated on the diagram as shown in Figure 3-9, where at the angle shown of 130° cannot meet the negative value of the phase-shifted sine wave.

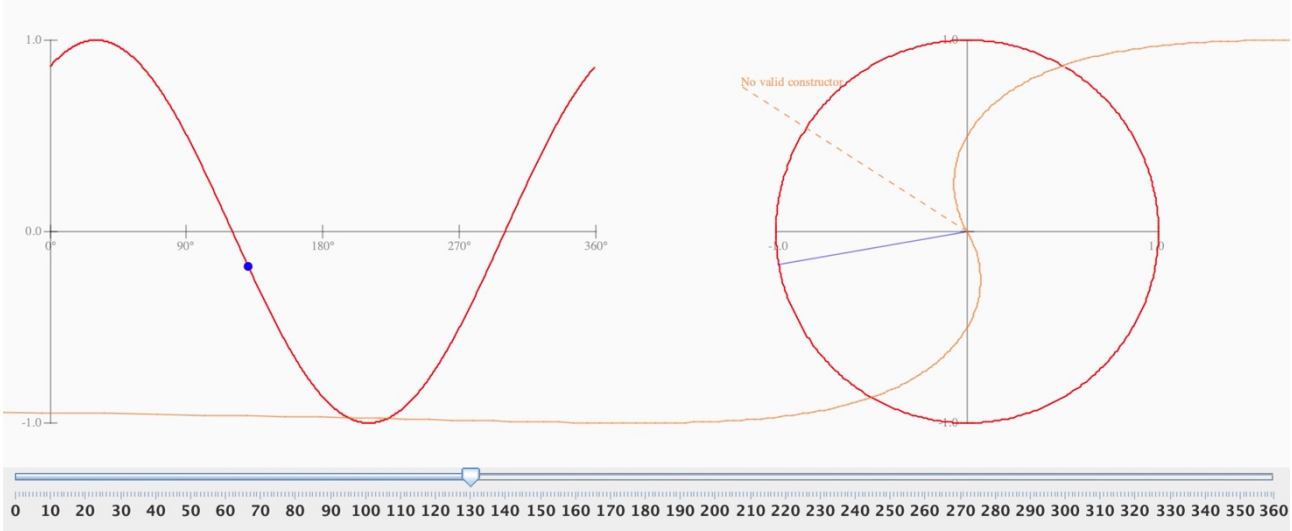


Figure 3-9 Example of Angle with no Valid Constructor

3.10 Difference between Reference Figures and Constructor Figures

For the fundamental alone the reference and constructor figures are the same. See Figure 3-10.

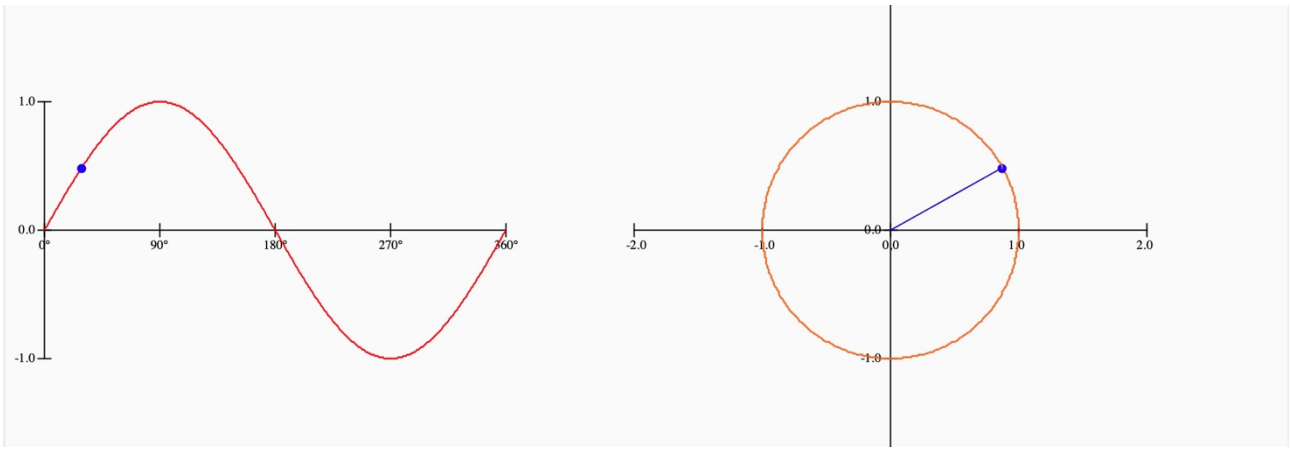


Figure 3-10 Reference and Constructor Figures for Fundamental

With the second harmonic (or first overtone) the shape is the same, but the constructor figure is translated 0.5 units to the right. See Figure 3-11. The proof is in appendix 14.

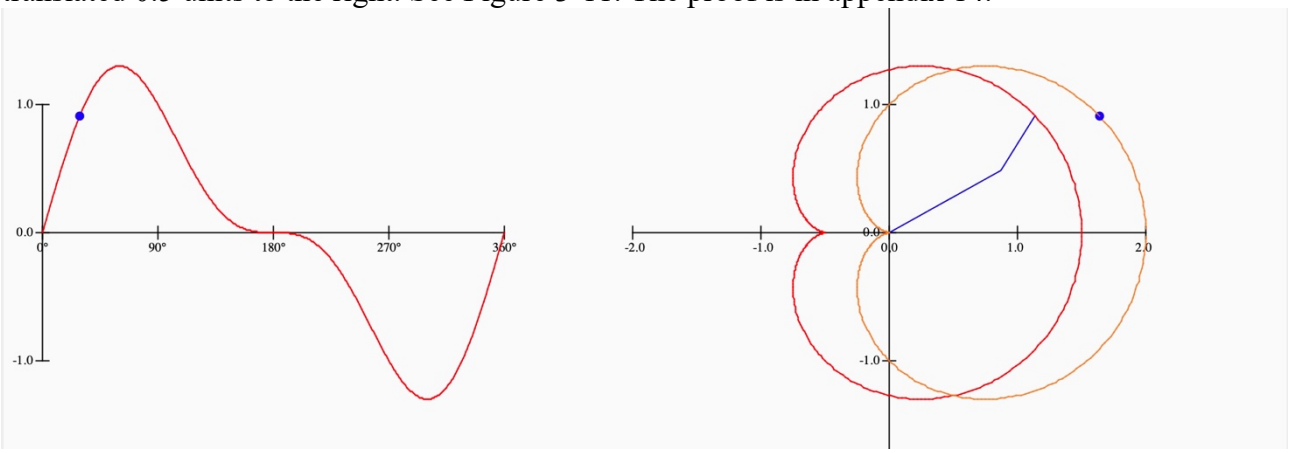


Figure 3-11 Fundamental and Second Harmonic with Reference (red) and Constructor (orange) Figures

From the third harmonic on, the constructor figure looks slightly different to the reference figure – see Figure 3-12.

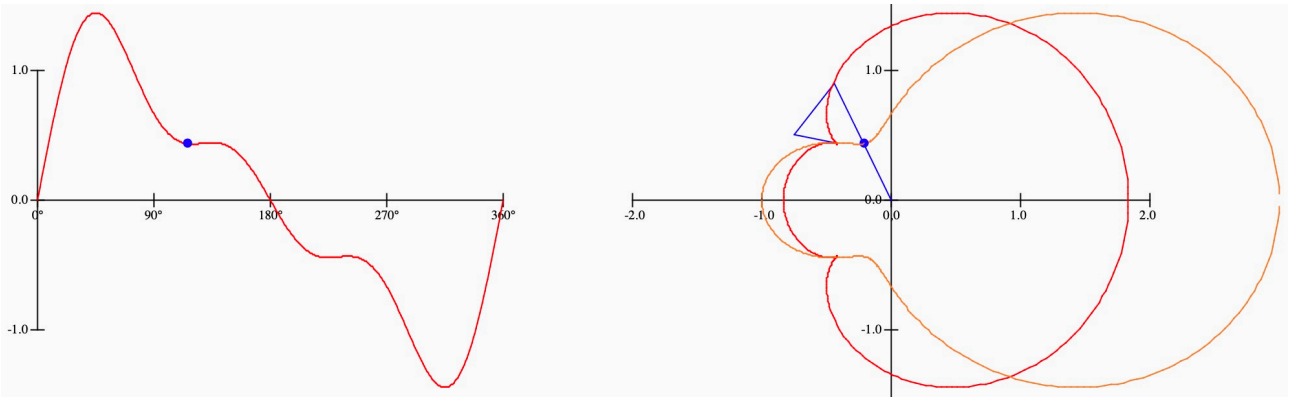


Figure 3-12 Fundamental, Second and Third Harmonic with Reference (red) and Constructor (orange) Figures

Using the visualisation program, we observe that as we pass through the angles of the cusps where the resultant wave is almost flat, the end of the vector lingers at the cusp, and the constructor must be nearly flat to correspond to the resultant wave – see Figure 3-13.

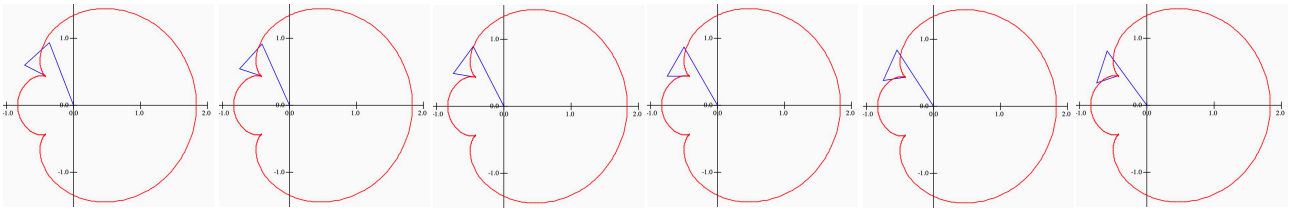


Figure 3-13 Vector passing through cusp around 120°

3.11 The Sound

The waves are synthesised with the overtones having relative frequencies to the fundamental. To play the sound, a wave must be produced with a particular frequency. A wave has a wavelength, which is the distance from 0° to 360°. The frequency is related to the wavelength by the speed of sound in the medium through which it is travelling (usually air):

$$speed = frequency * wavelength$$

For a vibrating string or pipe, the wavelength is fixed and determined by the length of the pipe or string[3] (see 3.3), so:

$$frequency = \frac{speed}{wavelength}$$

(This explains why inhaling Helium and speaking gives the voice a higher frequency – the speed of sound in Helium is about three times higher than in air.)

The program generates a sound at a selected frequency. (In modern concert pitch A is 440Hz, middle C is 263.18139Hz.)

The program allows the frequency to be set, and this could be used for waves other than sound waves to make them audible.

3.12 The WAV File

The sound can be saved as a WAV file (see 7.3.2). The file consists of digital samples of the calculated sound wave at the selected sampling frequency on one channel (mono, not stereo). The length of the file depends on the duration specified by the user, the default being 1 second. A frequency of 1Hz for 1 second at a sampling rate of 44000 samples/second will therefore give a WAV file with 44000 samples of one cycle. Doubling the frequency halves the number of samples per cycle. This in turn affects the frequency resolution of the Fourier transform that can be performed on the WAV file – see 3.13.

3.13 Analysis – the Fourier Transforms

There is plenty of literature about the Fourier transform, which transforms the representation of the wave as amplitude over time into a representation of amplitude against frequency, and the basics are not repeated here.

3.13.1 Transform Types

The analysis function in the program uses either a simple Discrete Fourier Transform (DFT) or a Fast Fourier Transform (FFT) or a Chirp Z-Transform (CZT). DFT is slower than FFT but more flexible in the choice of sample window – the FFT needs a window whose length is a power of 2. The CZT can effectively zoom in to a specific range of frequencies within the DFT. The time taken

[3] This is simplified – the frequency of a string also depends on its mass per unit of length and its tension.

for the DFT is proportional to n^2, the square of the number of samples in the window. The time taken for the FFT is proportional to nlog(n).

The following optimisations are implemented in the DFT to give the best results:

- The algorithm is based on an input of only real values (rather than complex).
- Only the left half of the symmetric double-sided result is calculated and shown (symmetry about the Nyquist frequency). The amplitude values are therefore doubled as the total energy is usually divided between the two symmetric frequencies.
- The arctan function is very sensitive to small values, so phases may not be accurate.
- Averaging and STFT calculations are distributed over the available processors of the computer hardware.

The FFT is a faster algorithm but requires that the number of samples in the window is a power of 2 (…, 64, 128, 256, 512, …). This also only uses real values; the imaginary values of the complex numbers all being set to 0.

The Chirp Z-Transform, implemented here in its simple form (not actually using chirp signals[4]), calculates the Discrete Fourier Transform over a given range of frequencies with a given step (delta) between the frequencies. Note that this does not increase the effective resolution, but zooms in on a part of the spectrum, and will distribute the amplitude of a DFT frequency bin over the neighbouring bins.
It is not possible to increase the frequency resolution without losing resolution in time. Taking a longer window will give better frequency resolution, but if the frequency is not constant, more frequencies will be present in the time window and less will be known about when the frequencies occurred. This is a kind of uncertainty principle: the better the frequency resolution, the worse the time resolution and vice-versa.
A further method is to use wavelets which use variable windows to optimise the time vs. frequency trade-off. This has not (yet) been implemented in the program.

3.13.2 DC

The frequency spectrum x-axis starts at 0Hz which is the DC (direct current) component of the wave if the amplitude is not symmetrical about 0 amplitude.

The following are worth bearing in mind:

- The DC component is the mean over the period.
- The integral over the period p is the same for any starting point t_0, so integrating over 0 *to* p is the same as from t_0 *to* $p+t_0$. In other words, it is not necessary to set the marks at the zero crossings of the wave, but this will naturally give different phase results.
- The human ear is phase deaf and cannot perceive differences in the phases of the harmonics (Ohm's law of acoustics). For audible acoustic waves the phases can be ignored by setting all the phases to zero in the synthesis.

3.13.3 Windowing Functions

Various windowing functions are provided. These are useful when the window being analysed is not a whole number of cycles or the sampling rate is not divisible by the frequency (there are not two samples exactly 360° apart).

[4] Computers have advanced in processing power since the 1990's such that it is not necessary to use the fast algorithm of Bluestein or Rabiner with of convolutions with chirps by forward and inverse FFTs. The complexity of the programming is not justified by the gain in speed.

3.13.4 Short Time Fourier Transform STFT

A Short Time Fourier Transform (STFT) is used to analyse the spectrum over time rather than at a single point in time. The result is usually displayed as a spectrogram (an example is in Figure 3-14).

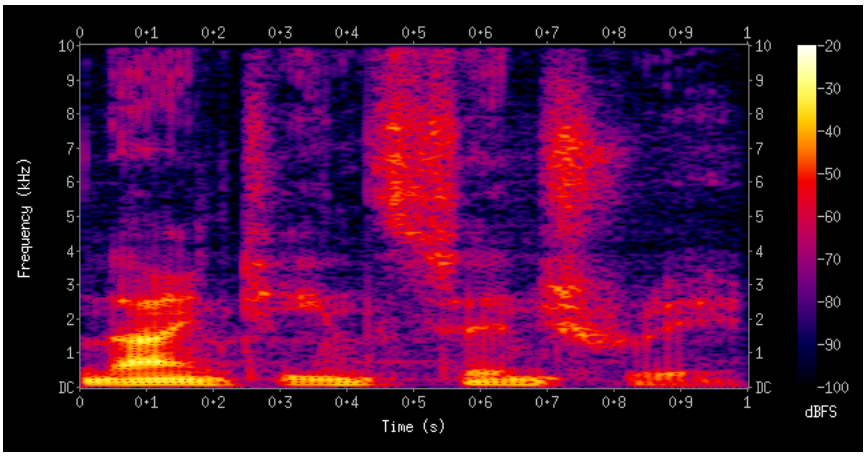


Figure 3-14 Spectrogram By Aquegg - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=5544473

These have the frequency on the y-axis and time on the x-axis. The colour represents the amplitude of each frequency band at each point in time. However, these are often difficult to interpret.
Therefore, the Fourier Round Trip program displays the spectrum as an animation over time: when the animation is played the spectrum changes, the speed can be controlled or the animation stepped from one time unit to the next. The position of the current frame in the input wave is shown by a cursor in the input panel, similarly to when the sound is being played.
The animation also includes the cumulative line spectrum if this is selected for viewing.

3.13.5 Resolution and Bins

The resolution of the Fourier transform depends on the sampling rate and the number of samples in the window selected from the wave to transform. The spectrum is divided into bins, each representing a band of frequencies given by the resolution. The amplitude of the spectral frequency represented by a bin contains the contribution of all the frequencies of the bin.

3.13.6 Leakage, Peak Detection and Interpolation

If the window used for the spectrum resulting from the Fourier Transform does not cover an exact number of cycles, as is generally the case with real-world signals, there will be leakage between the bins of the spectrum. The spectral line of a frequency which would fall between the bins is "smeared" between adjacent bins. There are other aliasing effects that can cause frequencies further from the spectral line to be added to it.
To obtain a more accurate frequency from an area of leakage, a curve can be fitted to the points around a peak. This is done in two steps:

1. Peak detection
2. Interpolation

There are very many methods for peak detection with varying levels of complexity, but since here the emphasis is on musical signals, a simple form of automatic peak detection is provided with the option to manually correct the estimates – the human eye and brain is the best peak detector, and this enables analysis of particular points of interest.
There are also various algorithms for interpolation to give an estimate of the true frequency and amplitude of a line in a leaky spectrum.

4 Possible Uses

4.1 Sounds

The most obvious and most frequent use is to analyse sound data to determine its spectrum.

4.2 Planetary Orbits

To show the orbit of another planet as seen from Earth, the orbit of that planet must be combined with that of Earth. Approximating the orbits with circles rather than ellipses can do this. The resulting reference figure shows the required orbit. The program is designed to show one cycle, but by multiplying the frequencies by a common factor, multiple cycles can be shown, limited in resolution by the number of points calculated.

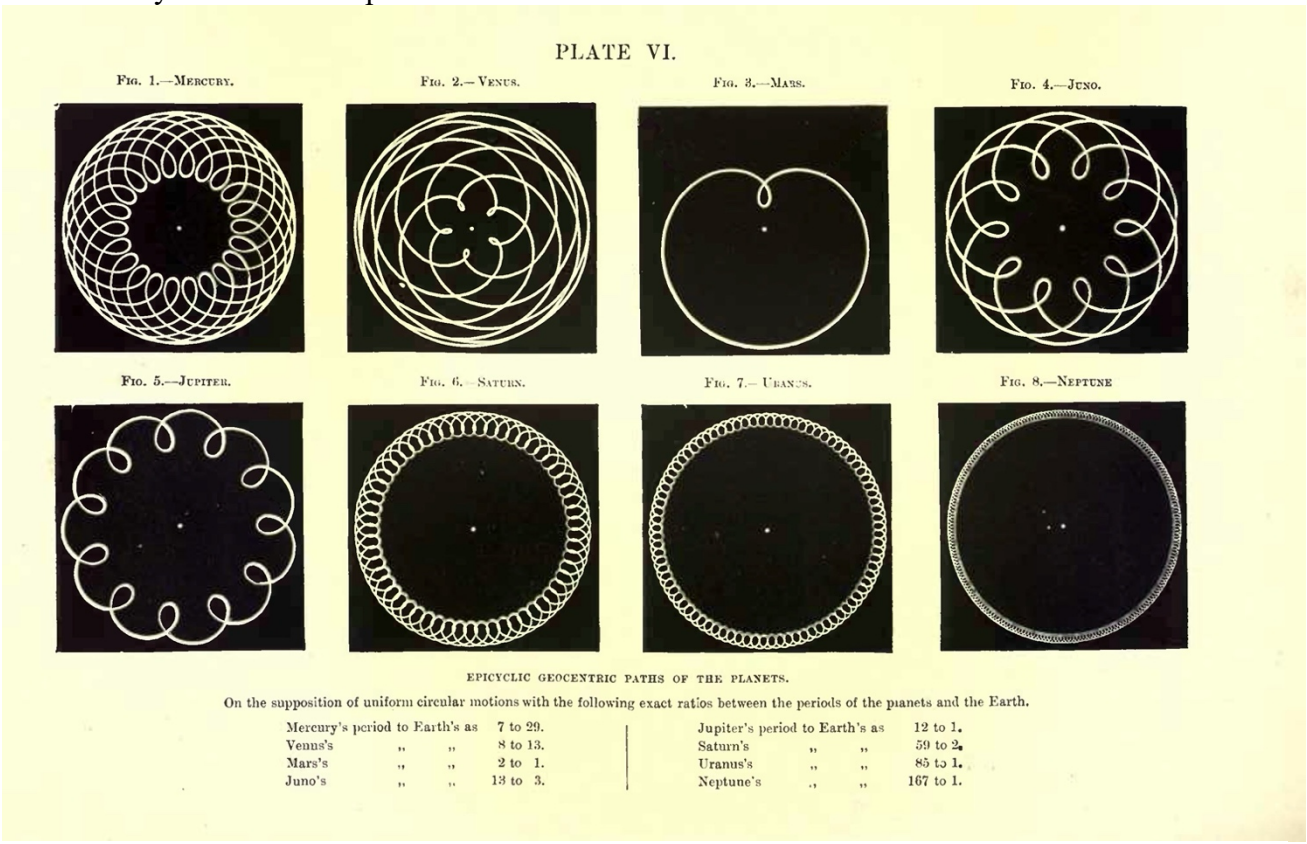

PLATE VI.

Fig. 1.—Mercury. Fig. 2.—Venus. Fig. 3.—Mars. Fig. 4.—Juno.

Fig. 5.—Jupiter. Fig. 6.—Saturn. Fig. 7.—Uranus. Fig. 8.—Neptune

EPICYCLIC GEOCENTRIC PATHS OF THE PLANETS.

On the supposition of uniform circular motions with the following exact ratios between the periods of the planets and the Earth.

Mercury's period to Earth's as	7 to 29.	Jupiter's period to Earth's as	12 to 1.
Venus's ,, ,,	8 to 13.	Saturn's ,, ,,	59 to 2.
Mars's ,, ,,	2 to 1.	Uranus's ,, ,,	85 to 1.
Juno's ,, ,,	13 to 3.	Neptune's ,, ,,	167 to 1.

Figure 4-1 Some Planetary Orbits as Seen from Earth. Source: Old and New Astronomy. Proctor, Richard Antony. 1892. Longmans London. Between pp. 164&165.
https://archive.org/details/oldnewastronomy00procuoft/page/n185/mode/2up

The commonly shown diagrams in Figure 4-1 can be reproduced by converting the orbit radii as amplitudes relative to Earth's orbital radius and the frequencies with suitable factors in relation to Earth's orbit frequency as shown in Table 4-1 (decimal is comma) – this is with more modern data from http://gerdbreitenbach.de/planet/planet.html.

Table 4-1 Conversion of Planetary Orbits as Waves

	R rel Earth	Period T	Frequency 1/T	1/R	Factor	Best Freq. 000	Best Ampl. 000	Best Freq. 001	Best Ampl. 001
Earth	1	1	1,00000	1,00000					
Mercury	0,387	0,24084	4,15213	2,58398	8	8,00000	1,00000	33,21707	0,38700
Venus	0,723	0,61519	1,62551	1,38313	10	10,00000	1,00000	16,25514	0,72300
Mars	1,524	1,88082	0,53168	0,65617	2	2,00000	1,00000	1,06337	1,52400
Jupiter	5,203	11,862	0,08430	0,19220	15	15,00000	1,00000	1,26454	5,20300

	R rel Earth	Period T	Frequency 1/T	1/R	Factor	Best Freq. 000	Best Ampl. 000	Best Freq. 001	Best Ampl. 001
Saturn	9,539	29,458	0,03395	0,10483	30	30,00000	1,00000	1,01840	9,53900
Uranus	19,201	84,014	0,01190	0,05208	50	50,00000	1,00000	0,59514	19,20100
Neptune	30,047	164,793	0,00607	0,03328	50	50,00000	1,00000	0,30341	30,04700
Pluto	39,482	247,94	0,00403	0,02533	50	50,00000	1,00000	0,20166	39,48200

For example, Mars is shown in Figure 4-3 with the above data, which does not quite join up after one cycle.

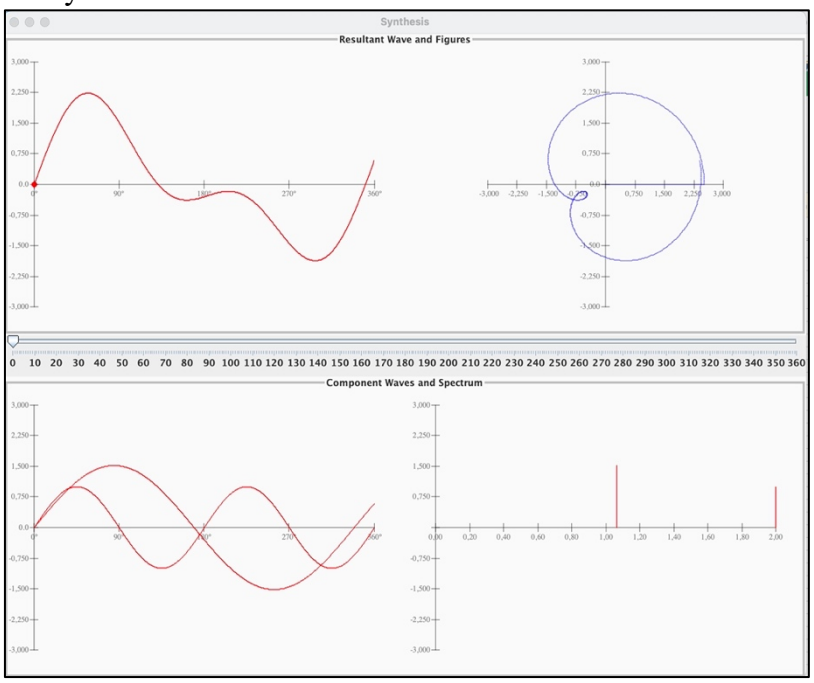


Figure 4-3 Relative Orbit of Mars as Seen from Earth – Modern Data

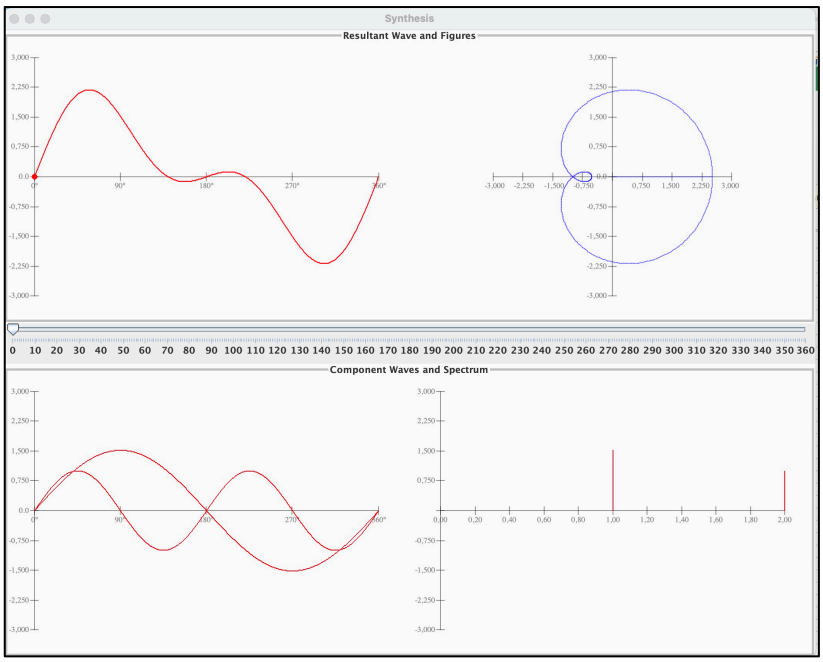


Figure 4-2 Relative Oribits of Mars Seen from Earth - Proctor's Data

Using the frequency ratio given by Proctor in Figure 4-1 we can reproduce his image – see Figure 4-2.

4.3 Musical Temperament

Musical intervals can be shown by combining frequencies in the appropriate ratio.
Figure 4-4 shows the unison 1:1, that is two equal waves giving a sine wave with amplitude 2.

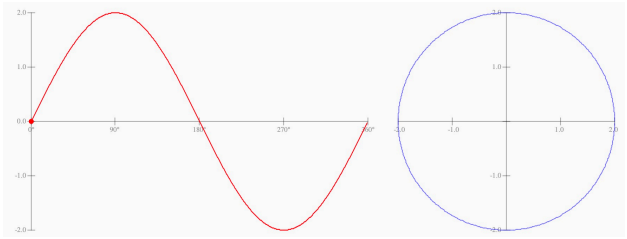


Figure 4-4 Unison

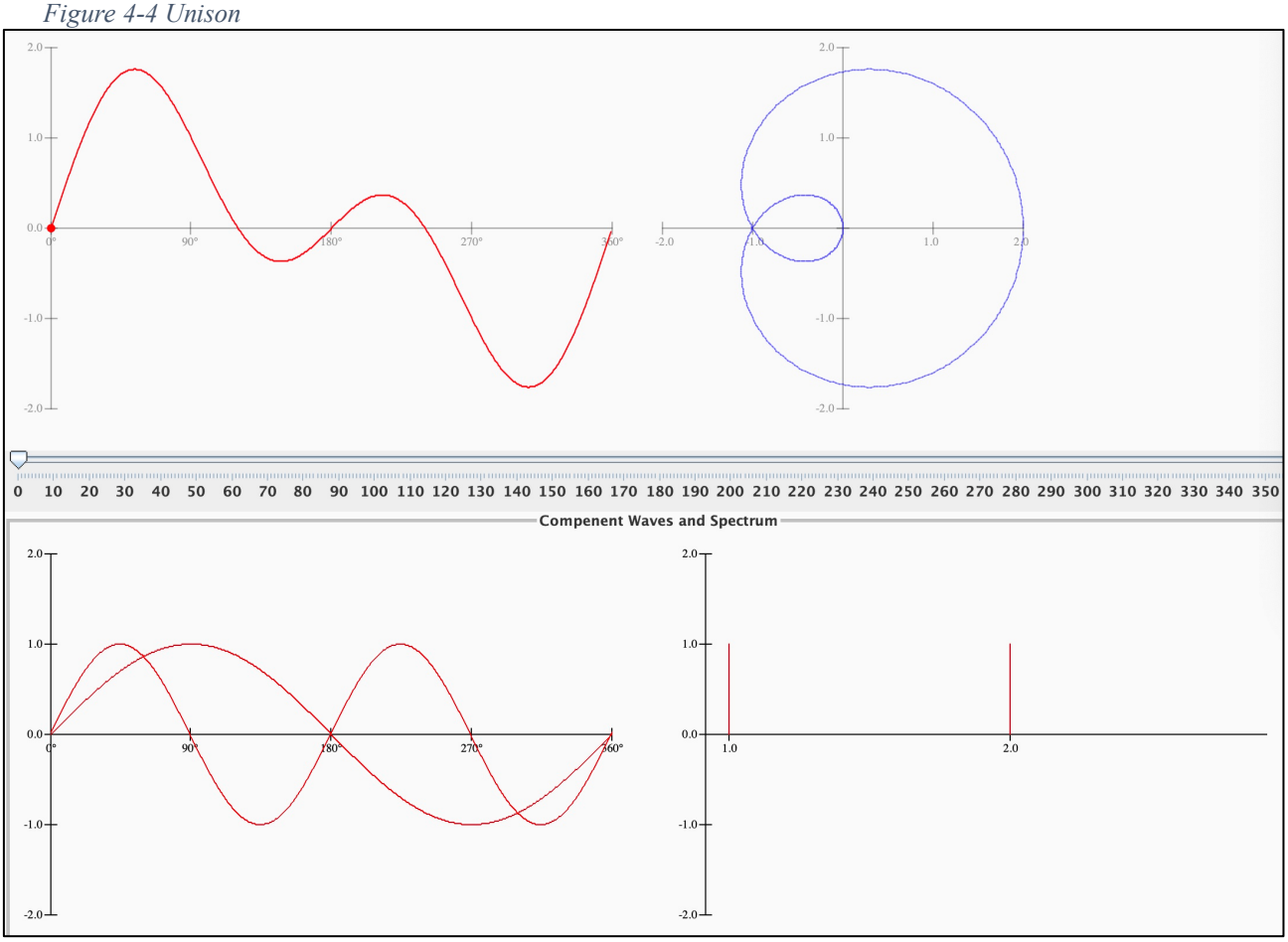


Figure 4-5 Octave

Figure 4-5 shows the octave 1:2 with the second wave twice the frequency of the first and the same amplitude.

Figure 4-6 shows a perfect fifth or 2:3 as 1: 1.5.

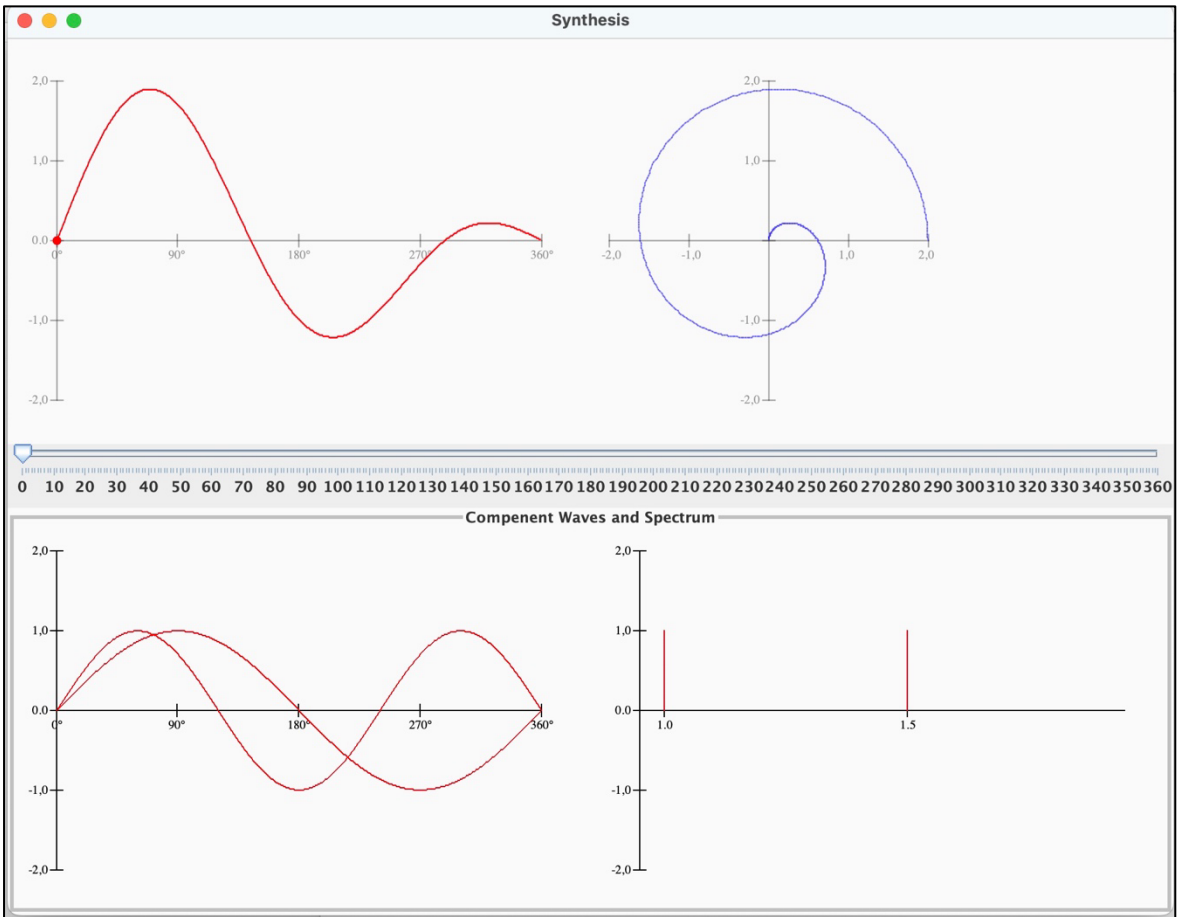


Figure 4-6 Perfect Fifth

Figure 4-7 shows a “wolf” fifth that occurs with meantone temperaments, a ratio of 1:1.46 instead of 1:1.5. Comparing this with Figure 4-6, the second wave, and thus the resultant wave, does not return to 0 at 360° and the reference figure does not reach the origin.

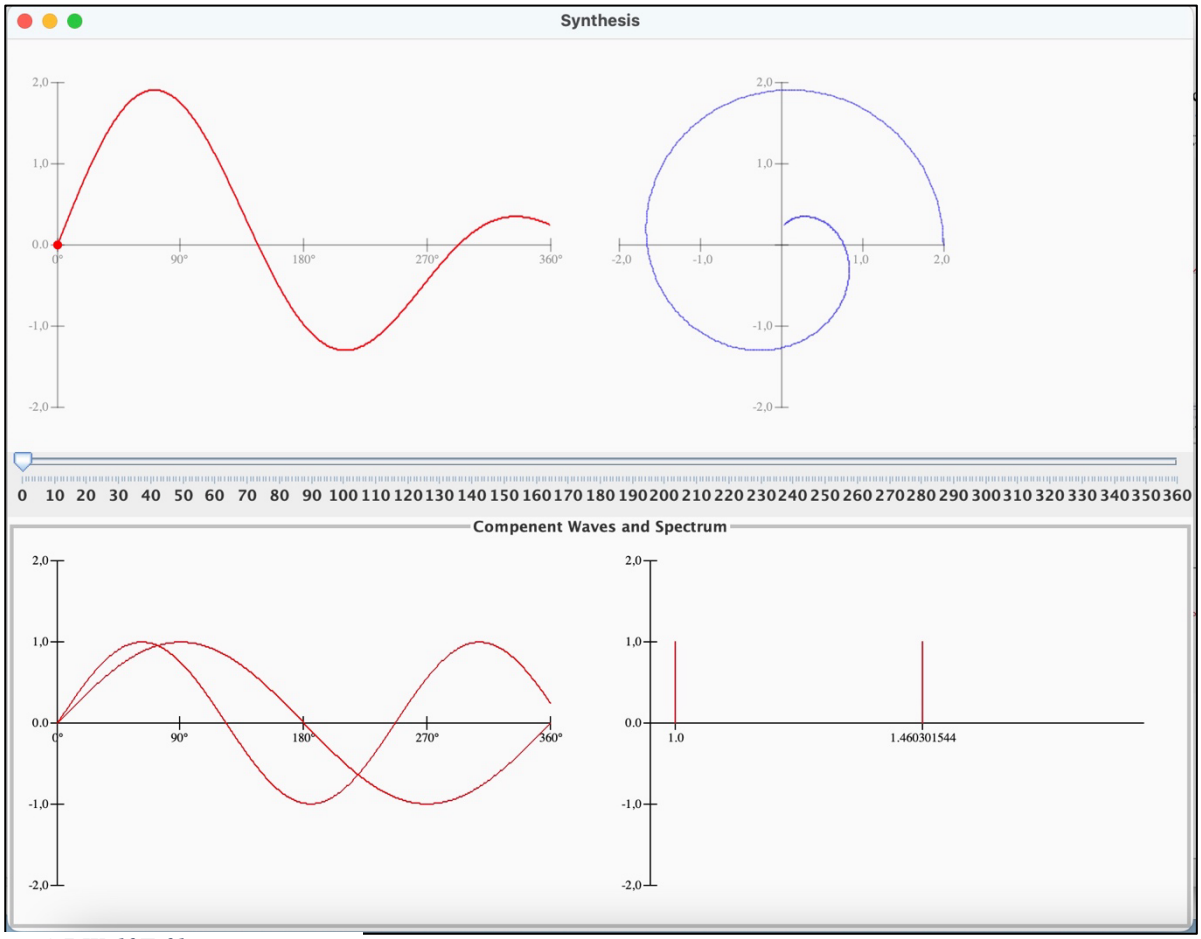


Figure 4-7 Wolf Fifth

Further aspects of temperament can be studied in the analysis spectrum, where the notes of the scale and the harmonics can be superimposed on the spectrum.

4.4 Drawing

The reference figure can be used to construct almost any line drawing from the epicycles that form the reference figure. There are examples on the Internet, e.g.
https://contra.medium.com/drawing-anything-with-fourier-series-using-blender-and-python-c0881e1b738c
https://www.jezzamon.com/fourier/index.html
https://blog.wolfram.com/2013/05/17/making-formulas-for-everything-from-pi-to-the-pink-panther-to-sir-isaac-newton/
https://www.wolframalpha.com/input/?i=butterfly+curve
https://www.youtube.com/watch?v=qS4H6PEcCCA

This one (Figure 4-8) was constructed by modifying https://www.jezzamon.com/fourier/index.html to extract the component wave parameters.

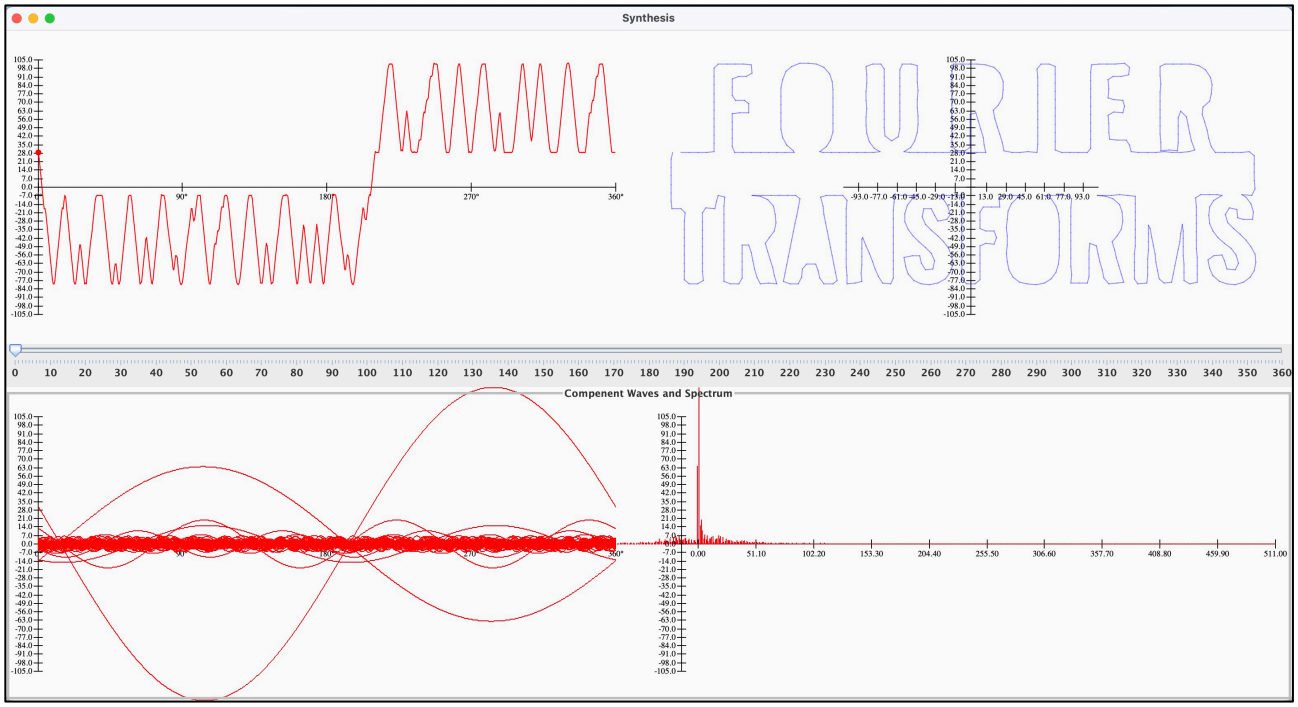


Figure 4-8 Drawing from waves

4.5 Seismology

4.6 Oceans and Lakes

4.6.1 Tides

4.6.2 Waves

4.6.3 Hydrophones

4.7 Gravitational Waves

See [GEO600], [LIGO], [Heinzel].

5 Installing the Program

5.1 Download

The program itself can be downloaded from:
https://github.com/Goldberg53/harmonics.git
with no guarantees that it will work as expected.

5.2 Prerequisite

The program is written in the Java programming language. The Java runtime environment (JRE) must be installed before the program can run.
Obtain the official free JRE from Oracle at www.java.com and install it as instructed.

5.3 Caveats

The program has been tested, but as with any software, there can be no guarantee that all results are correct, and no responsibility is taken for any harm caused by using the program.

The following differences between the systems are known:

- Apple MacOS has tighter security restrictions on downloading and running unsigned files. The program file is not signed, so the user must confirm download and execution.

5.4 Installing the Program

The program does not need to be installed as an application. Download the .jar file to a convenient place on your computer. You can create a shortcut to the jar file and copy the shortcut to the desktop.
This user manual is included in the .jar file and can be extracted and opened with the "Extract Manual" function in the Help menu. Further information is in the Help function in the Help menu.

5.5 Removing the Program

To remove the program from your computer, simply delete the .jar file.
If you extracted the user manual, you can delete the extracted file in your home folder, called Harmonics.pdf or similar at any time.
If desired, uninstall the Java runtime environment in the usual way e.g. on Windows with Programs and Features in the Control Panel, on Mac by deleting the files.

5.6 Error Handling

Error messages will appear in dialogue boxes on the screen. Some programming errors may crash the program with a stack dump to the Java console.
If the error message indicates insufficient memory in analysis, try using less samples between Mark1 and Mark2 or using a smaller sound input file.

6 Main Window

6.1 Menus

The **Program** Menu has:

Settings Currently only the colours can be configured with settings – see 6.1.1.

Close Terminates the program**.**

The **Help** menu has:

Release Notes shows the version history of the program with the changes made to each version.

About gives the origin and development information of the program.

Help gives assistance with extracting the manual.

Extract Manual will extract this user manual document from the .jar file and open it. It first tries to extract it to the directory in which the .jar file is located, and if that fails, tries the user's home directory. The location is shown in a confirmation message. The manual will then be opened in the PDF reader. If any of these operations fails, an appropriate message is given – see Help for alternatives.

6.1.1 Settings – Colours



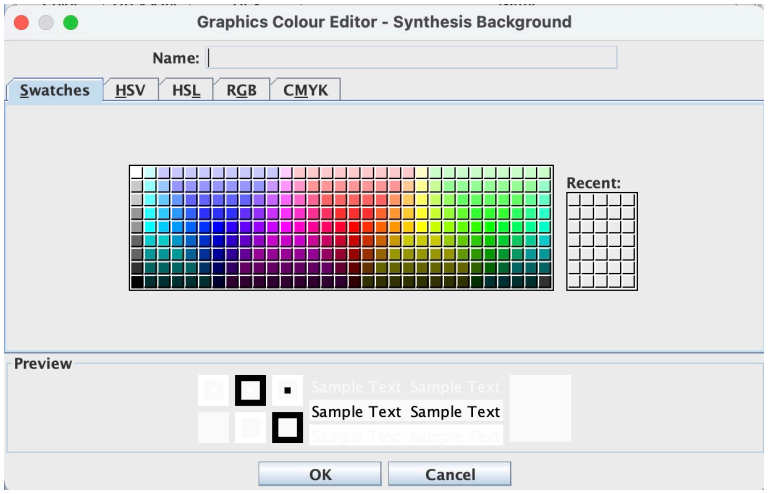
Color	GreyScale	Hex	Name
		0xfffcfcfc	Synthesis Background
		0xcffa90b	Analysis Background
		0xb5000000	Axis Color
		0x3c000000	Axis Graticule Color
		0xffff0000	Plot Color
		0xffff0000	Wave Color
		0x780000ff	Reference Figure Color
		0xb4ed7d31	Constructor Figure Color
		0xff00cc33	Reference Figure Vector Color
		0x61009900	Reference Figure Circle Color
		0xb4ed7d31	Reference Figure Point Color
		0xffff0000	Constructor Figure Point Color
		0x78a0b4a0	Sound Color
		0x7800ff00	Sound Window Color
		0x80ff0000	Mark1 Color
		0xffff0000	Mark1 Label Color
		0x800000ff	Mark2 Color
		0xff0000ff	Mark2 Label Color
		0xb4ff8400	Padding Color
		0x640000ff	FFT Expansion Color
		0xffc88436	Audio Cursor Color
		0xffc81414	Video Cursor Color
		0xffff3232	Message Color
		0xb4000000	FT Window Color
		0xb4006464	FT Overlap Color
		0xb4ff8400	FT Final Color
		0xffff0000	FT Stale Color
		0xb4009600	Average Wave Color
		0xb4b400ff	Decay Interpolation Color
		0xffb400ff	Decay Interpolation Color Full
		0x9a0000ff	Spectrum Notes Color
		0xd6007800	Spectrum Harmonics Color
		0xffb400ff	Spectrum Peaks Color
		0xff0000ff	Spectrum Interpolations Color
		0xff000000	Spectrum Cumulative Color

The colours of various items in the graphics can be configured according to taste or visual necessity.
The colours are as follows (column S = Synthesis, A = Analysis):

Colour Name	Usage	S	A
Synthesis Background	Background of synthesis plots	x	
Analysis Background	Background of analysis plots		x
Axis Color	Axes of all the plots	x	x
Axis Graticule Color	Horizontal lines at pi in phase plot		x
Plot Color	Synthesis: Plots of component waves and spectral lines, spectral and phase lines. Analysis: Plot of transform in window function details pop-ups	x	x
Wave Color	The resultant wave	x	
Reference Figure Color	The reference figure	x	
Constructur Figure Color	The constructor figure	x	
Reference Figure Vector Color	The vectors of the reference figure	x	
Reference Figure Circle Color	The circles of the reference figure	x	
Reference Figure Point Color	The point corresponding to the current angle from the reference figure on the resultant wave	x	
Constructor Figure Point Color	The point corresponding to the current angle on the constructor figure	x	
Sound Color	The input wave in the Average and Decay and Input Wave plots		x
Sound Window Color	The part of the input wave between the marks in the Average and Decay and Input Wave plots		x
Mark1 Color	The colour of the vertical line of the first mark in the Average and Decay and Input Wave plots		x
Mark1 Label Color	The colour of the label of the first mark in the Average and Decay and Input Wave plots		x
Mark2 Color	The colour of the vertical line of the second mark in the Average and Decay and Input Wave plots		x
Mark2 Label Color	The colour of the label of the second mark in the Average and Decay and Input Wave plots		x
Padding Color	The horizontal line at showing the zero padding in the Input Wave plot, and a vertical mark at it end		x
FFT Expansion Color	The horizontal line showing the zero padding to a power of 2 fo the FFT		x
Audio Cursor Color	The cursor showing the position of the sound being played in the Input Wave plot		
Video Cursor Color	The cursor showing the position of the current STFT frame in the Input Wave plot		x
Message Color	The message in resultant wave when drawing cancelled	x	
FT Window Color	The outline window function in the Input Wave and window function details pop-ups		x
FT Overlap Color	The outline window function of the frames of Averaged and STFT		x
FT Final Color	The input wave resulting from application of the window function, including the zero padding.		x
FT Stale Color	The stale indicator in the spectrum and phase plots		x
Average Wave Color	The averaged wave in the Average and Decay plot		x

Colour Name	Usage	S	A
Decay Interpolation Color	The interpolation result of the decay in the Average and Decay plot		x
Decay Interpolation Color Full	The interpolation result text in the Average and Decay plot		x
Spectrum Notes Color	The musical notes overlaying the spectrum		x
Spectrum Harmonics Color	The harmonics overlaying the spectrum		x
Spectrum Peaks Color	The crosses marking the peaks in the spectrum		x
Spectrum Interpolations Color	The interpolated frequencies in the spectrum		x
Spectrum Cumulative Color	The cumulative line spectrum		x

Double-click on a colour row to open the colour editor:



There are tabs for various colour models which can be used to change the colour. The result is previewed at the bottom of the window.
The buttons below the colour table in Settings are:

Export

Exports the colour settings to a file on the user's disc. The file contains the names and hexadecimal colour codes in a comma separated variable (CSV) file. The format is described in Appendix 11.

Import

Imports the colour settings from a previously exported file.

Reset to Defaults

Reset all the colours to their original default values.

Close

Close the window

6.2 Tabs

The main window has two tabs:

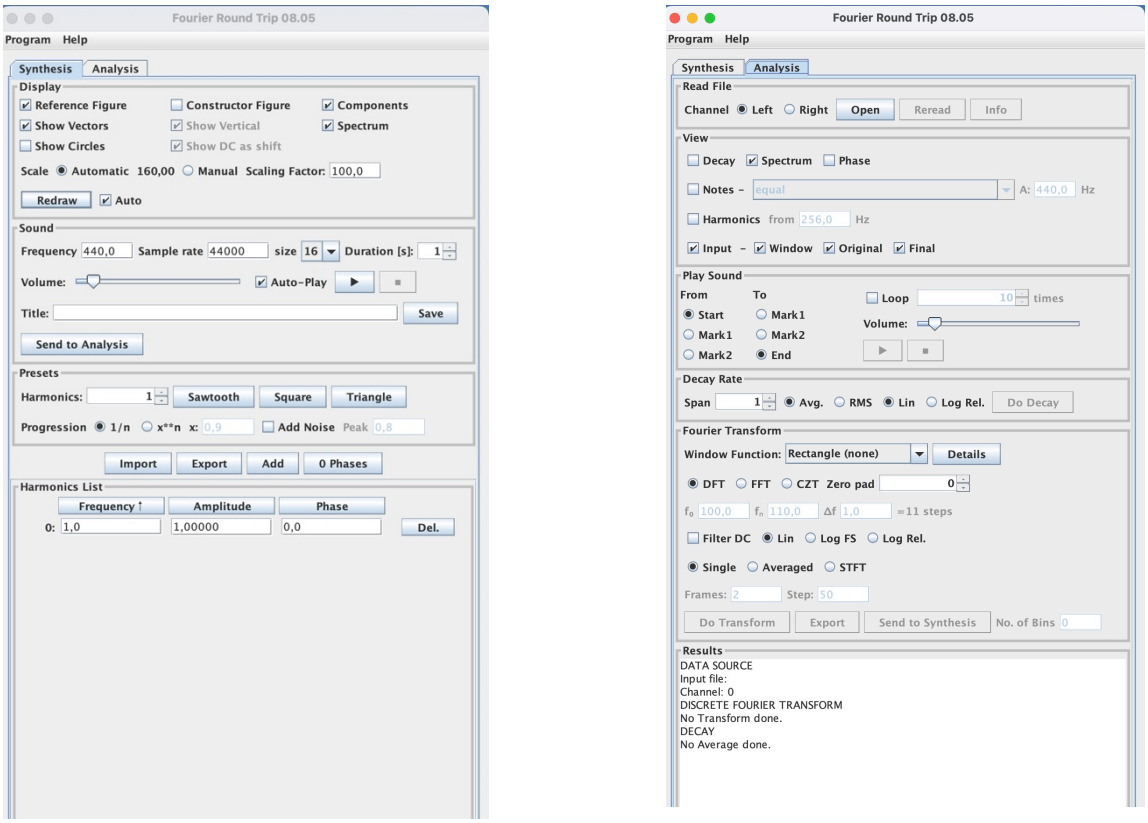


Figure 6-1 Tabs of Main Window

The left tab contains the settings for the synthesis, the right tab those for analysis.

7 Synthesis

7.1 Notes

DC

To make the visualisation correspond to the use of sine waves starting with an amplitude of 0 at 0 degrees, we use sine wave throughout. The more usual convention for Fourier analysis is to use cosine waves because this allows DC components to be included – cos(0) = 1.
We therefore adopt the convention of treating a zero frequency as DC for the sine waves.
See also 3.9.

7.2 Synthesis Display

A new synthesis window is created with the **New** button and shows the results depending on the parameters set in the main window. It is updated with the **Redraw** button.

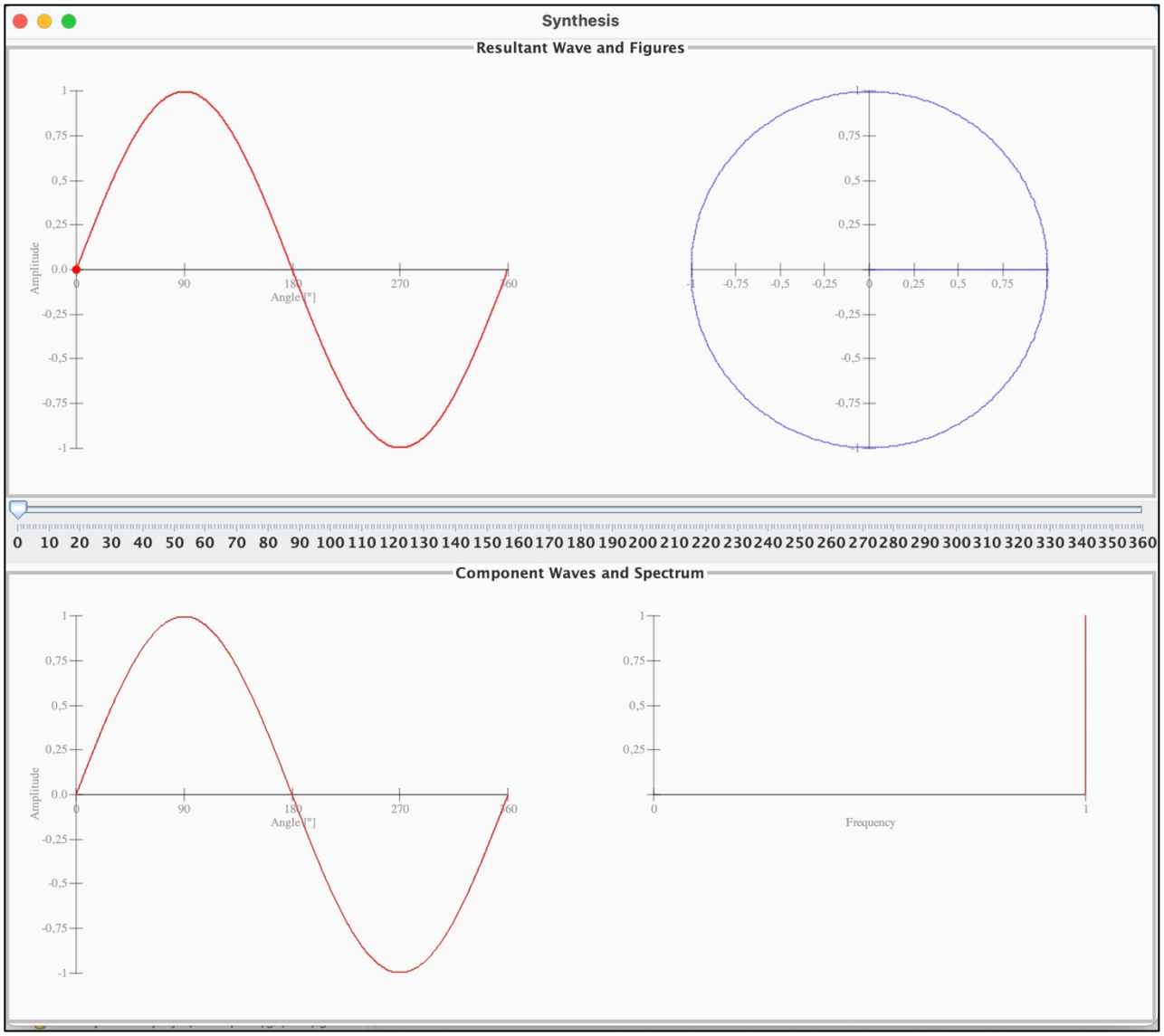


Figure 7-1 Synthesis Window

In the top half, the resultant wave on the left and optionally the reference and constructor figures superimposed on each other on the right.
The slider moves a blue dot along the resultant wave, shows a vector representation of the components on the reference figure in blue and an orange dot on the constructor figure with a dotted

orange line at the corresponding angle. All these are for the angle selected with the slider. For angles where no constructor is possible, the segments of the constructor figure are omitted.
The bottom half shows the component waves superimposed on each other on the left and the frequency spectrum on the right.
Note: the graphics display one cycle of the wave and do not include the decay.

Axes and Scaling
The resultant wave (top left) and the component wave (bottom left) diagrams are scaled vertically to accommodate the maximum amplitude. They are calibrated such that the end of the axis is the next round number equal to or above the maximum needed. They are then divided into a fixed number of divisions. Any numbers that overlap others are omitted.
The wave diagrams are scaled horizontally to take up half of the available window size and calibrated 0° to 360°.
The reference and constructor figures (top right) are scaled to match the amplitude and axis calibration of the waves vertically and with the same scaling and calibration horizontally. The diagram is positioned to be centred at ¾ of the window width and does not rescale with the window size in order to preserve the correct aspect ratio and to give some leeway in accommodating wider diagrams.
The frequency spectrum (bottom right) has a vertical axis corresponding to the waves as above. The horizontal frequency axis is given ten ticks up to the maximum frequency. As above, numbers are omitted if they would overlap. The y-axis is placed over the zero frequency, also when there are negative frequencies.

7.3 Synthesis Parameters

These are in the left tab of the main window – see Figure 7-2

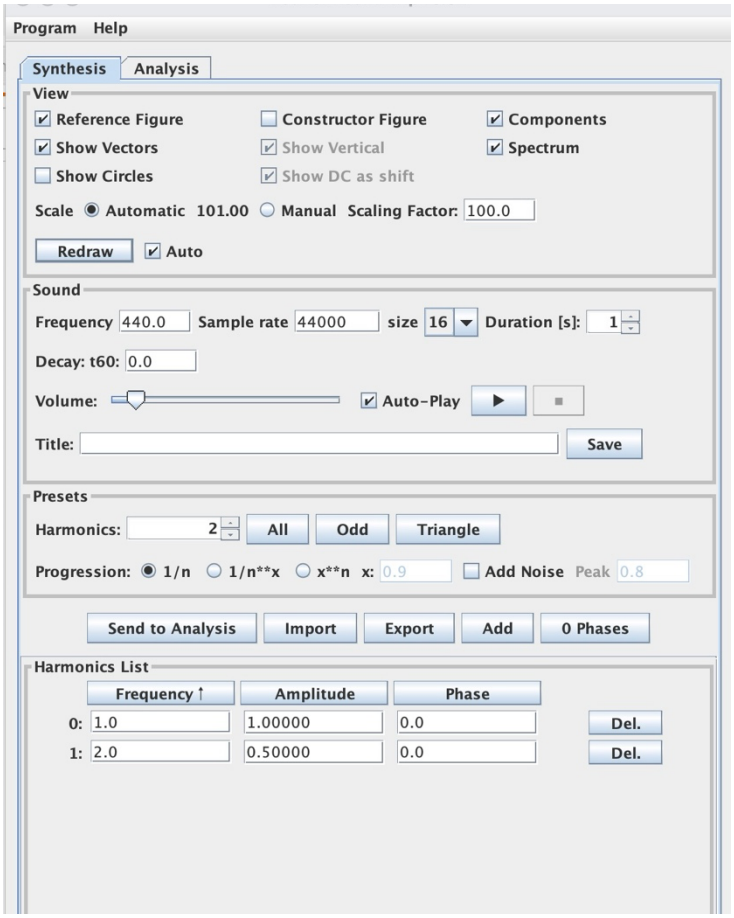
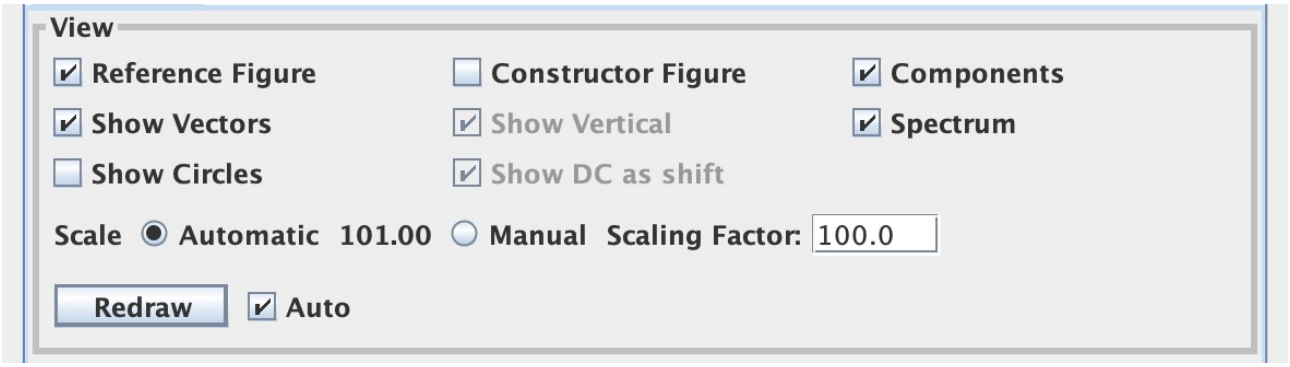


Figure 7-2 Synthesis Parameters Window

7.3.1 View Section



This section determines whether the reference and constructor figures are displayed.

Reference Figure

Check the box to display the reference figure.

Show Vectors

Check the box to display the vectors of the harmonics in the reference figure.

Show Circles

Check the box to display a circle for each harmonic. The circle traces the path of the end of the vector.

Constructor Figure

Check the box to display the constructor figure.

Components
Check the box to display the component waves.

Spectrum
Check the box to display the spectrum.

Show Vertical
Check the box to add the vertical line for the y-value and a horizontal line from there to the wave.

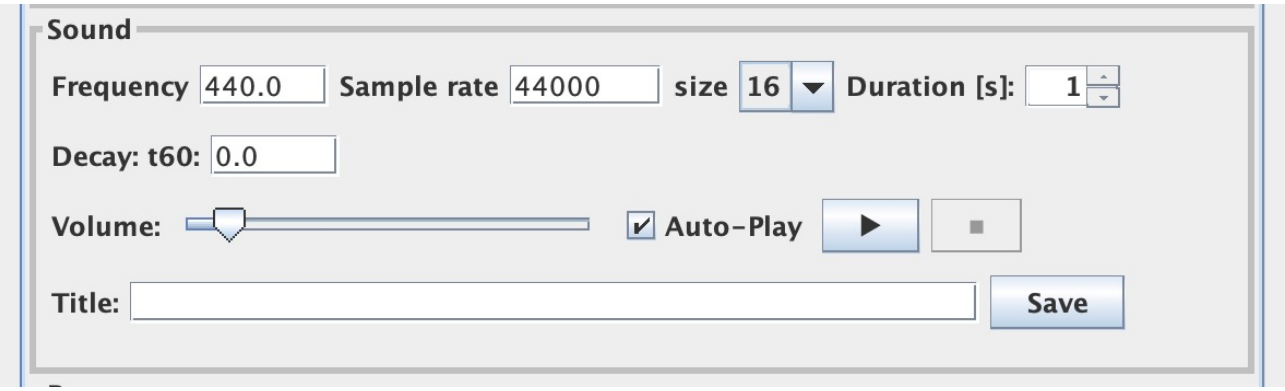
Show DC as shift
Check the box to move the origin of the constructor vertically for any DC (0 frequency) component. This will help in some cases where the constructor figure is incomplete. See section 3.8 and 3.9.

Scaling Factor
This is the scaling factor to make the graphics fit on the screen. Note that the automatic scaling is optimised for the height of the wave and the reference figure and may need to be scaled manually in some cases, in particular for the sideways direction of the reference and constructor figures, which can become very large.
When set to **Automatic**, the calculated scaling factor is shown on the right.
If **Manual** is selected, the value entered in the field will be used.

Redraw
Click this button to redraw the graphics. Depending on the number of harmonics, the calculation of the waves and the sound may take some time, so in that case a progress window is shown. The calculation can be cancelled.

Auto
If the box is checked the graphic will be redrawn automatically whenever a parameter is changed. The sound will also be played if "Play on Redraw" is selected in the Sound section (a sound already being played will be interrupted first).
Note: redraw does not occur when the number of harmonics or the progression is changed as harmonics may have been entered manually in the list.

7.3.2 Sound Section



These control the sounding of the resultant wave played via the computer's audio output and sent to the analysis.

Frequency
The frequency of the note to be played.

Sample Rate
The sample rate in samples per second to be used to write the WAV file. It is best to choose a rate that is divisible by the most important frequency to be measured, and for Fast Fourier Transform FFT), a power of 2.

Sample Size
The number of bits in each sound sample for playing the sound and sending it to the analysis function. Note: this does not affect the saved sound file.

Duration
How many seconds the note will play when triggered, and how long the sound sent to analysis will be.

Decay t60
The decay time specified by the number of seconds the sound will take to decay to -60dB below the initial amplitude. A value of 0.0 gives no decay. See Appendix 15.

Volume
Controls the volume of the note played.
Note: the volume is also affected by the volume setting on the computer itself and the connected amplifier and speakers.
NOTE: the volume control does not work at sample sizes above 16 bits, and these play at full volume. Reduce the computer's volume control before playing!

Auto-Play
The sound will be generated and the note played whenever the graphic is redrawn or any of the frequency, sample rate or duration are changed.

Title
Enter a one-line title (of effectively unlimited length). This is included as meta-data for the Track Title when the sound file is written with Save and in the CSV export of the Harmonics List.

Save
Click this button to save the sound as a .wav file. A file chooser will appear for selection of the file location and name. The file is saved as a single-channel (monophonic) sound sampled at the set sample rate, sample size 16 bits, with the duration given above. This can be read by the analysis function – see chapter 8.
It includes meta-data for the track title (see above) and comments with the harmonics list (see 7.3.5 below). These can be seen after reading into the analysis with the Info button - see section 8.6.1.

Play►
Click this button to play the sound at any time.

Stop■
Click this button to stop the sound playing (useful if you have selected a long duration).

Sent to Analysis
Send the generated sound to the analysis function.

7.3.3 Presets Section



The buttons will create various wave forms with the selected number of harmonics. The constructed values of frequency, amplitude and phase are populated into the Harmonics List below. These can be used as a basis for further adjustment of the individual harmonics.

Harmonics
Select the number of harmonics to be populated by the other buttons.

All
Creates all the harmonics, e.g. for a sawtooth wave.

Odd
Creates only the odd harmonics e.g. for a square wave.

Triangle

Creates the harmonics for a triangular wave.

Progression

1/n or **1/n^x** or **x^n**

The progression of the amplitudes of the harmonics can be selected either as the usual 1/n (n is the number of the harmonic) or 1/n^x or with exponentials x^n – see section 3.4. The Triangle option does not use these.

x

The value of x can be entered in the field.

Note: for the 1/n and 1/n^2 progressions, n is the number of the harmonic and n=1 is the first harmonic or fundamental, i.e. index 0 in the table. For the x^n progression, n is the number of the partial or overtone starting at 1 and n=0 is the fundamental so that the fundamental is multiplied by x^0 = 1.

Note: the radio buttons do not redraw the graphic – it is assumed that other adjustments will be made first.

Add Noise

Tick this checkbox to add random noise to the generated wave.

Peak

Set the peak amplitude of the added noise.

7.3.4 Buttons



Import

Click this button to import the Harmonics List, Frequency, Sample Rate and Title from a CSV file (see 7.3.6).

Export

Click this button to export the Harmonics List, Frequency, Sample Rate and Title to a CSV file (see 7.3.6).

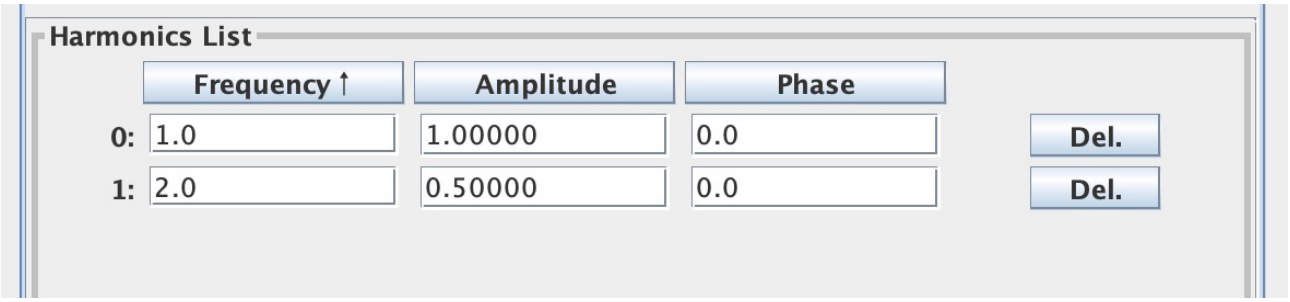
Add

Click this button to add another harmonic to the list. It will preset the frequency to the next integer above the last frequency in the list (which depends on the sorting of the list), preset the amplitude and the phase to 0. Note that it does not attempt to set the amplitude to the next in the progression, as it is assumed that this is for a manual setup. To add harmonics to a preset series, increase the number of harmonics and regenerate the waveform.

0 Phases

Click this button to set all the phases to zero. This is useful when the harmonics have been sent from an Analysis.

7.3.5 Harmonics List Section



This shows the harmonics being used. A list created from the Presets above can be modified. The list can be exported as a CSV file and imported from a previously exported file (see above)

Sorting

Click on the column header buttons to determine how the harmonics list will be sorted. The sorting occurs immediately when the button is clicked. The current sorting is shown with an arrow, up for ascending and down for descending on the sorted column.

This can be used to choose how the vectors in the reference figure are shown: sorting by ascending amplitude will plot the longest vectors first.

Frequency fields

Set the relative frequency of the harmonic as a multiple of the fundamental frequency. Frequency 0 is the DC (direct current) component and Frequency 1.00 is the fundamental frequency. The frequency of the sound is set in the sound section.

Amplitude fields

Set the relative amplitude of the harmonic.

Phase fields

Set the phase of the harmonic in degrees. A phase shift of 90° makes the sine into a cosine.

Delete

Click this button to remove the harmonic from the list.

7.3.6 Import and Export Format

The CSV file must start with the first line giving the CSV separator (comma or semicolon) – this can be read by Excel:

```
sep=,
```

or

```
sep=;
```

Line 2: `Description;` <Title>

Line 3: `Frequency;`<frequency>;<sample rate>

Line 4: `Noise;` <true/false>;<peak> for on/off and the peak (see section 7.3.3)

Line 5: column titles for the harmonics:

```
Frequency;Amplitude;Phase
```

followed by the data values

<frequency>;<amplitude>;<phase>

Other parameters are not saved as they are likely to be set for a session independently of the individual data files.

For export the CSV separator is chosen as follows:

If the system language has comma as the decimal separator, then the CSV separator is semicolon.

Otherwise, the CSV separator is comma.

The thousands and decimal separators are according to the language set on the system. Conversion may be necessary if files written in English are to be read in German, French, etc. or vice-versa.

Example:

```
sep=;
Description;100 Hz with 3 sawtooth harmonics
Frequency;100,0;44000
Noise;false;0,5
Frequency;Amplitude;Phase
1,000000;1,000000;0,000000
2,000000;0,500000;0,000000
3,000000;0,333333;0,000000
```

Note: the imported file or data sent from analysis can be a spectrum of any wave, so the entries in the harmonics list may not actually be harmonics of the fundamental frequency, but are the frequency bins of the analysed spectrum.

8 Analysis

8.1 Overview

8.1.1 WAV files

The following data codings are currently supported for reading from WAV files:

1 or 2 channels
8-, 12-, 16-, 24-, 32-, 64-bits per sample
Any sample rate
PCM unsigned 8-bit is always unsigned)
PCM signed (12-, 16-, 24-, 32- and 64[5]-bits per sample (>8 bits are always signed)
PCM floating point 32-bit and 64-bit
A-Law, μ-Law 8-bit (these are dynamic range compression methods for telephony)
Little endian (No big-endian files have been found so far.)

There are other programs which can convert other file formats to a usable WAV format, for example the Apple AIFF format can be converted in Mac OS with the terminal command:

```
afconvert -f WAVE -d LEI16 <inputfile>.aiff /<outputfile>.wav
```

Floating point input file formats are converted to 24-bit signed for further processing.

8.1.2 Round Trip

As stated in the synthesis section 7.1, we use sine waves rather than cosine waves throughout. This means that the phase spectrum has an alteration of +90° to give sine waves in the result. When passed back to the synthesis in a round trip they then correspond to the original wave.

8.1.3 Window-Functions

Some formulae for the window-functions give results with varying peaks. All these window-functions have been normalised to have a peak of 1.0.

8.1.4 Analysis Window

The analysis window appears when a wav file is opened or is sent from the synthesis and has four sections,

- The **Average and Decay** panel shows the input wave and performs the average calculation between the marks and the interpolated decay rate.
- The **Input Wave** showing the wave read from a file or passed from the synthesis and is used to set up the Fourier Transform.
- The **Fourier Transform Frequency Spectrum** showing the result of the Fourier Transform. The y-axis is calibrated with the amplitude values and can be set to linear or logarithmic. The x-axis is the relative frequency of the spectral components (bins), with a second x-axis for the corresponding frequencies in Hz (derived from the sampling rate of the input file). The red vertical lines show the component frequencies with their amplitudes.
- The **Fourier Transform Phase Spectrum** showing the corresponding phases on the same relative frequency x-axis as the frequency spectrum. The y-axis is calibrated in radians +/- π.

Below the frequency and phase diagrams are:

- a scroll bar and magnification slider. These operate on the frequency axis of both the frequency and the phase spectra.
- A set of buttons to set the magnification and scroll to commonly used settings.

[5] 64-bit PCM files are reduced to 32-bit to keep the values manageable.

If a Short Time Fourier Transform is selected, a panel of buttons for controlling the animated display of frames is below the spectrum.
Each section can be hidden by unchecking the corresponding “Show” button in the Analysis Parameters window to give more vertical space to the other sections.
The file name of the file read in is shown in the title bar.
The status lines of the average and input panels show the current magnification of the time axis in samples per pixel, the size of the window selected for analysis in samples and in milliseconds, and (for the input wave) the effective fundamental frequency given by the selected window assuming the window covers one cycle; the absolute frequency can be seen from the middle Hz x-axis of the spectrum.
The colours are set in the Settings – see section 6.1.1.

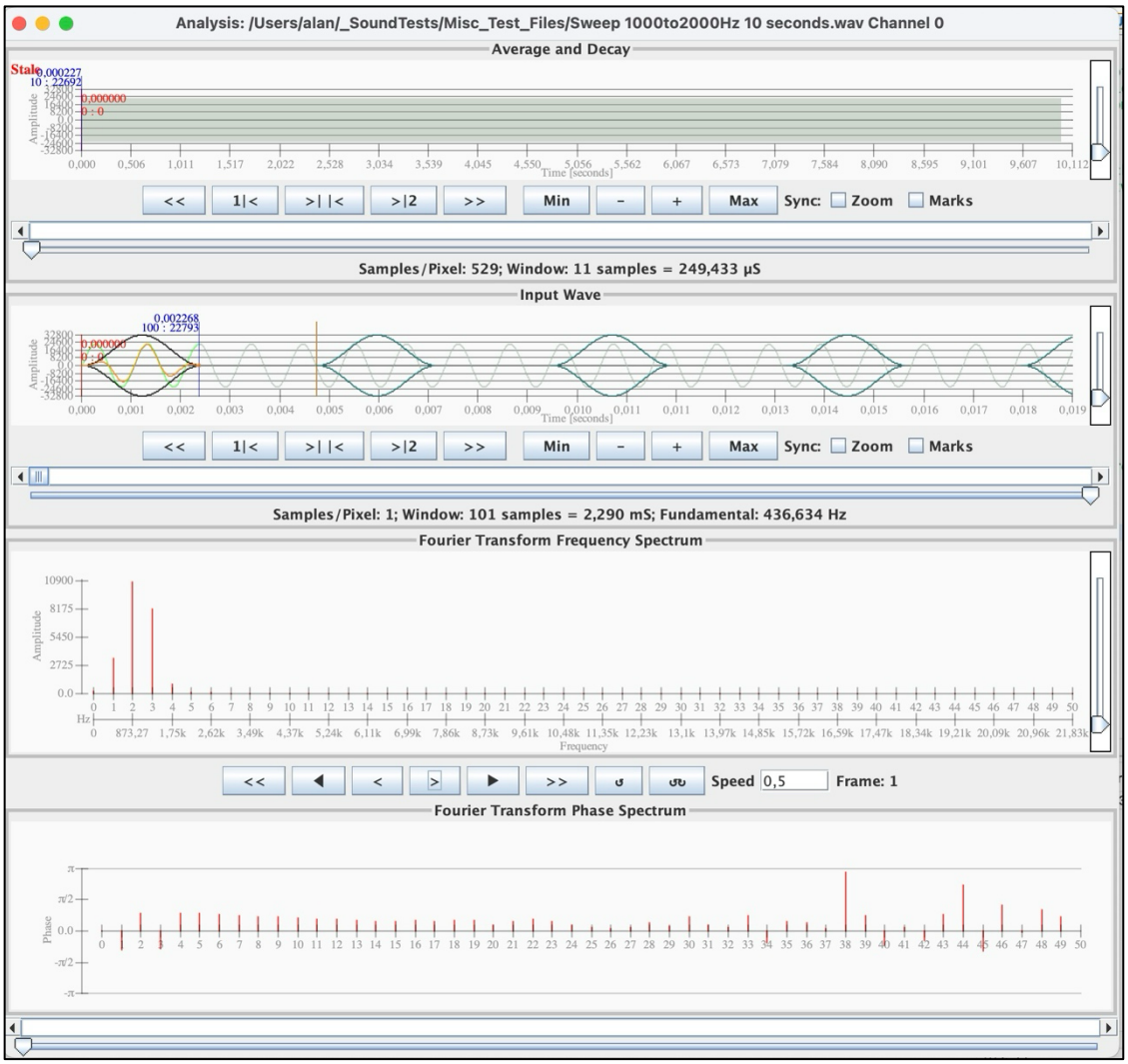


Figure 8-1 Overview of Analysis Window

8.1.5 Stale Indicators

If any parameters are changed, marks moved, etc., a “Stale” indicator is shown at the top left of the plots. This shows that the displayed results are not those of the current settings, but the previous result can still be viewed.

8.2 Input Wave Display

To start the analysis of a wave file, click New and select a file. The file is read and displayed in a new window. The Input Panel shows the complete wave file that has been read in. The window title gives the path of the file, or "from Synthesis" if the data was sent from the synthesis function (see 7.3.2 "Send to Analysis").

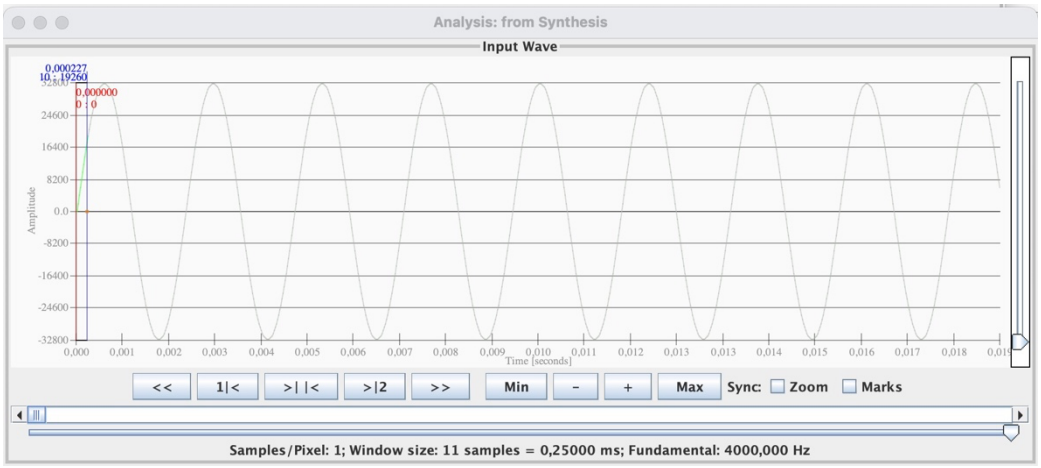


Figure 8-2 Analysis Input Diagram

The Y-axis is calibrated with the absolute values of the samples, e.g. -127 to +127.
The X-axis shows the time in seconds (to three decimal places giving milliseconds).

8.2.1 Markers

There are two coloured markers which define the window to be analysed, Mark1 on the left and Mark2 on the right. The wave within and outside the marks is coloured differently – see 6.1.1.
They can be moved as follows:

- Right click with the mouse at a point in the wave and select Mark1 or Mark2 to place it at that point.
- Drag one of the marks left or right with the mouse.
- Dragging with the shift-key pressed moves both marks together.
- A mark can be selected, either by click on the line or clicking on the label, and moved with the left or right arrow keys on the keyboard. Each key-press moves by one sample, so at magnifications below the maximum (1 sample per pixel), the mark might not visibly move. The shift-key acts as above, moving both marks together.

The positions of the markers are restricted such that:

- Mark1 must always be before (to the left of) Mark2.
- Neither marker can be moved beyond the start or end of the wave file. Note that when moving both marks with the shift-key, the movement will be blocked if one mark, which may not be visible, reaches the start or end of the file.
- The marks cannot be moved while an analysis is running.

The label above each marker shows the time in seconds to six decimal places (micro-seconds) and the sample number starting from 0 below that, followed by a colon and the amplitude value of the wave at that point.
Note: selecting the mark by clicking on the line can cause it to move inadvertently. For precise positioning, click on the label to select it and then use the left and right arrow keys.
Note that an excessive number of samples between the marks can cause the program to run out of memory. It will indicate this with an error message before exiting.

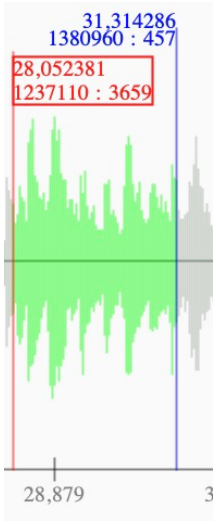


Figure 8-3 Marks for Analysis Window

For the best analysis results both markers should be placed at points with the same amplitude value for one cycle of the wave. This is often not possible as there may not be an actual sample at that point. This is best done at maximum zoom.

Note: the segment of the wave analysed (the analysis window) excludes the sample at Mark2. If Mark1 is at sample 0 and Mark2 is at sample 100, the window is 100 samples long from sample 0 to sample 99. The window size is shown in the status bar at the bottom of the analysis window and in the results summary in the Analysis tab of the main window.

8.2.2 Zero-Padding Marker

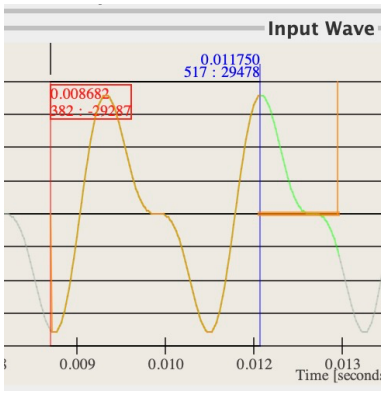


Figure 8-4 Zero Padding and Marker

If zero-padding is added to the window. A third marker above the x-axis and a thick line extending the final windowed wave shows its position – see 6.1.1 for colours. This may be beyond the end of the file, in which case it is not shown.

8.2.3 FFT Expansion

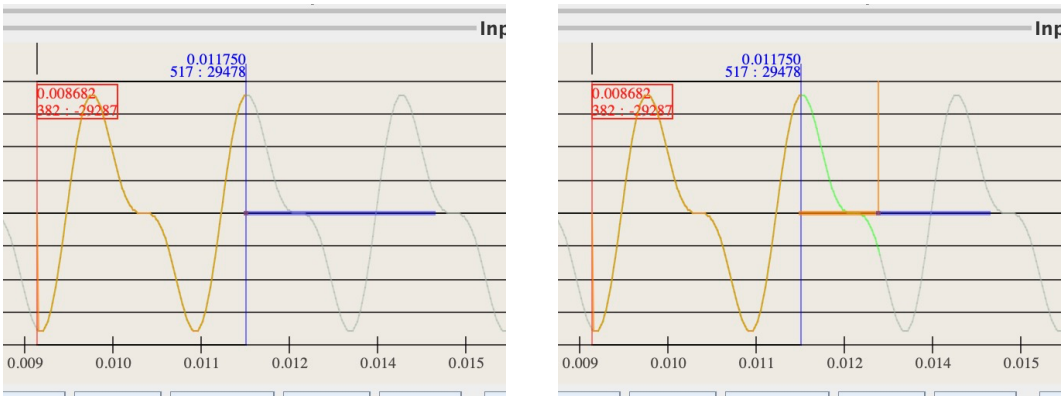


Figure 8-5 FFT Expansion, FFT Expansion with Padding

If the analysis window has been expanded to a power of 2 for a Fast Fourier Transform, this is shown as a line extending to the right of mark2 or to the right of the padding.

Note: the expansion to a power of 2 adds zeros before and after the analysis window. The line shown serves to give an impression of the number of samples added, e.g. to see what size of DFT would give the same resolution.

Note: this is only shown after the transform has been performed and is not stale.

8.2.4 Cursor

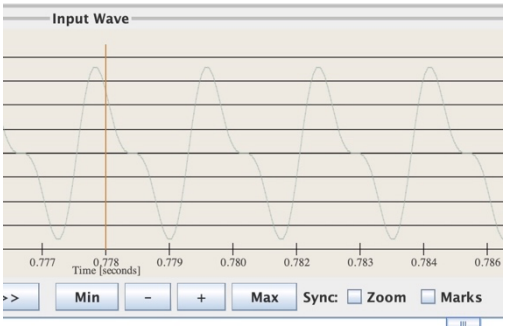


Figure 8-6 Cursor in Analysis Window

When playing the sound, a cursor shows the current position in the input wave – see 6.1.1 for colours. When showing the frames of a STFT, the cursor is at the start of the frame currently displayed.

When playing the sound is paused the cursor can be moved with the mouse to select the position from which to continue – see section 8.6.3.

8.2.5 Analysis Window and Final Windowed Wave

The shape of the analysis window and the final windowed wave are shown in brown see 6.1.1 for colours. These can be turned on or off with the checkboxes (section 8.6.2).

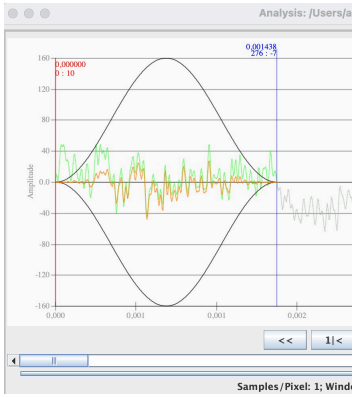


Figure 8-7 Analysis Window and Windowed Wave

Note: the final windowed wave is only shown after the transform has been done.

8.2.6 Multiple Frames

Multiple frames can be analysed, either to obtain the averaged spectrum of multiple frames or for a Short Time Fourier Transform. Averaging can be used for noise reduction, and the frames are normally chosen to overlap. The percentage overlap is given in the results pane (see section 8.6.10) and is negative if the frames have a gap between them.

The number of frames and the number of samples between frames (step) can be set by the user. The additional windows are shown with feinter lines in the graphic.

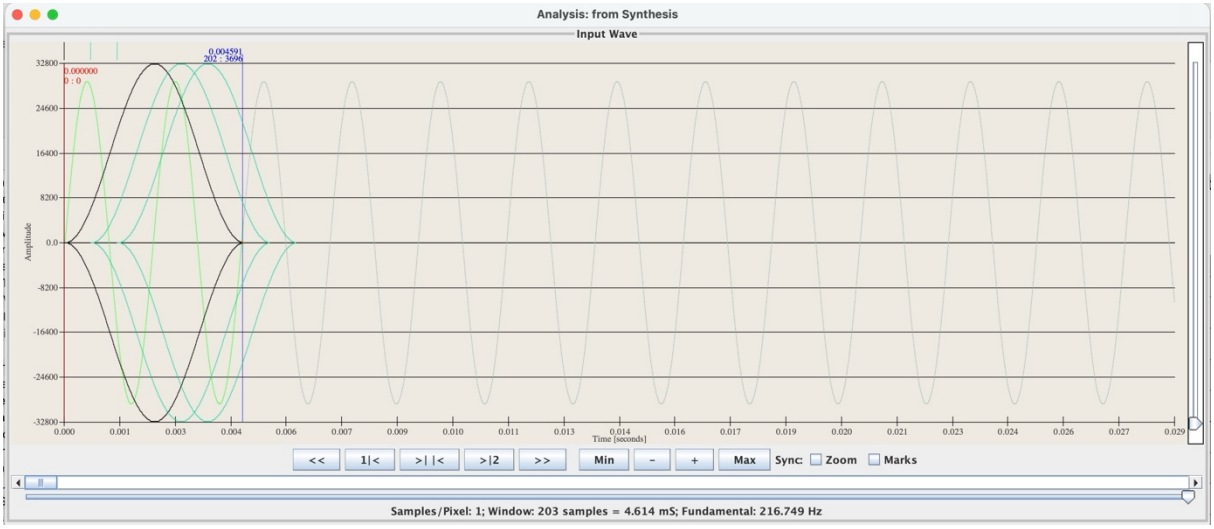


Figure 8-8 Overlapping Windows

Additional small tick marks are shown above the plot for the start of each window. This is particularly useful for the rectangular window, which is otherwise not easily visible.

8.2.7 Time Position and Zoom Buttons

Below the wave there are buttons for rapid positioning and zooming of the wave plot:



Figure 8-9 Position and Zoom Buttons

<< Go to start of the wave
1|< Go to Mark 1
>| |< Set the zoom and pan to show the entire window. This will put Mark1 on the left of the plot and set the magnification such that Mark2 is on the graph. Since the magnification is in integer values of samples per pixel, it will use less than the whole width if the next magnification level would put the end of the window outside the plot. Note that resizing the window does not maintain the entire window in the view.
>|2 Go to Mark 2
>> Go to end of the wave
Min Set minimum zoom to show the entire file in the width of the plot
- Zoom out by one step of the slider
+ Zoom in by one step of the slider
Max Set maximum zoom, one sample per pixel
The Sync checkboxes are used to synchronise the zoom/pan and the marks between the Average and Decay panel and the Input Wave panel – see section 8.4.

8.2.8 Time Axis Zoom and Scroll



Figure 8-10 Zoom and Scroll Sliders

At the bottom of this section there is a slider for the magnification, and above that, a scroll bar. The zoom slider has maximum zoom at the right at one sample per pixel and minimum zoom on the left which shows the entire file in the width of the screen. The slider moves from the maximum in steps of 1 from 1-5 and thereafter linearly to the value for minimum zoom. The current setting is shown in the status bar. Note that since the samples per pixel will not usually divide the total number of samples exactly, the last pixel may contain more or less samples than the given samples per pixel.

8.2.9 Amplitude Axes and Scaling

The wave from the input file is calibrated vertically according to the amplitude similarly to the waves in the synthesis window. The y-axis calibration is in the sampling units, e.g. -127 to +127. This can be magnified with the vertical slider on the right. The horizontal axis is the time in seconds (to three decimal places giving milliseconds).
The frequency spectrum gives the amplitudes on the y-axis, scaled linearly or logarithmically as selected by the radio buttons – see 8.6.5. This can be magnified with the vertical slider on the right. For logarithmic scales the slider raises or lowers the floor of the plot.
The phase spectrum (bottom) gives the phase in radians on the y-axis (π is 3.14) and the same frequencies as the frequency spectrum on the x-axis.

8.3 Average and Decay Display

8.3.1 Overview

This is similar to the input wave display and is used to calculate the decay rate between the marks; this is based on a running average amplitude of the waveform to avoid false results that would occur if a particularly high or low amplitude was used as the reference points.
Since most sounds decay in a logarithmic fashion, it is normally more useful to show the decay rate on a logarithmic scale in decibels. However, the logarithm of the wave values at the zero crossings cannot be calculated, as log(0) is minus infinity. The wave is therefore averaged from the absolute amplitudes of the samples. The number of samples either side of the current sample that are taken into the average of that sample is selected by the user and should normally be at least the number of samples in one cycle of the wave. This is called the span. To assist in selecting this the span is

shown as horizontal lines near the top of the mark lines (it may be invisible at lower horizontal magnifications).
The decay marks, as in the input wave display, show the time within the file and the sample number and sample value at the top of the mark. Additionally, the sample number and value of the average are shown at the bottom of the marks. Mark 2, the end of the averaging, also shows the difference from the average at mark 1 and the resulting decay rate, in decibels in dB/second for logarithmic scales. Note that this may be different from the result shown in the analysis results, which is derived from the linear interpolation. See 8.6.10.
The average curve and a linear interpolation are shown on the wave – see 6.1.1 for colours.

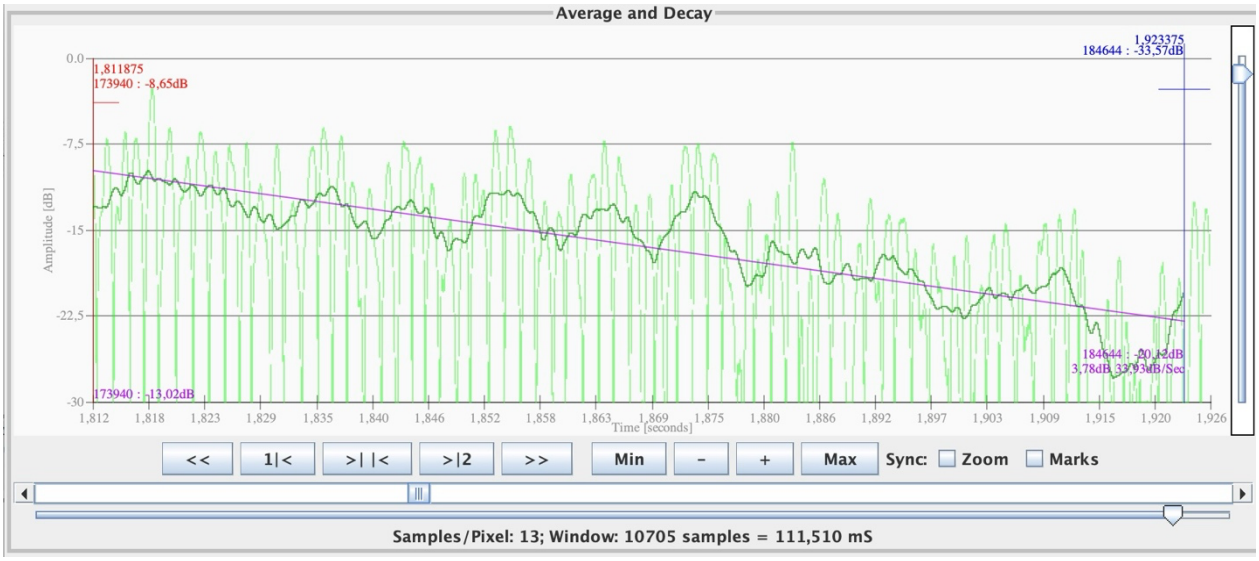


Figure 8-11 Average and Decay Display

The vertical zoom works differently in logarithmic mode – it effectively moves the floor upwards so that more detail is shown in the smaller dB values.

8.4 Synchronisation

The Average/Decay panel and the Input Wave panel can be synchronised.
Clicking the Sync Zoom button on one panel sets the zoom and pan of the other panel to match.
Clicking the Sync Marks button on one panel sets the marks of the other panel to match.
When Sync Zoom is ticked, moving the zoom or pan of one panel moves the other panel as well.
When Sync Marks is ticked, moving the marks in one panel moves them in the other panel as well.
The buttons act in the same way.
Some use cases are:

Analyse a Fourier transform and decay over the same sample window – check both Zoom and Marks.
Display the input wave on linear and log scales – check Zoom.
Analyse the Fourier transform of part of the decay curve – check Zoom but not Marks.

8.5 Spectrum Display

8.5.1 Overview

The spectrum shows a vertical line for each frequency bin. The x-axis is calibrated in bins, and the corresponding frequency in Hz on a second axis below this. The graph can be zoomed and panned together with the controls, which, if the phase plot is show, appear below that (they both zoom and pan together).

The y-axis calibration depends on the chosen scale, linear, logarithmic full scale or logarithmic relative to the maximum. The log scales are in decibels.
The y-axis magnification is controlled by the vertical slider on the right of the plot.
Whether the spectrum is shown and its content are determined in the View panel of the main Analysis Parameters window – see 8.6.2.

8.5.2 Cumulative Line Spectrum

The cumulative line spectrum can be superimposed. This shows the cumulative power (amplitude squared) as it is added by each harmonic on a scale from 0 to 100% with the scale on the right-hand side. See [Koenig] pp.8, 43, 322. This can be useful with a noisy signal – pure white noise will show a straight line from 0 to 100% of the spectrum.

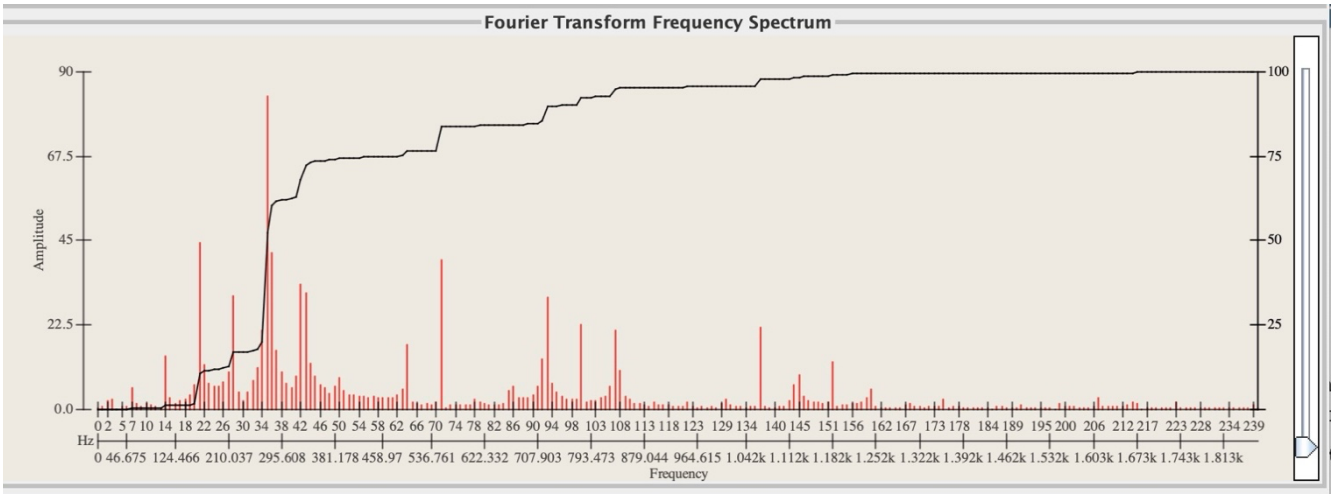


Figure 8-12 Spectrum Display with Cumulative Line Spectrum

8.5.3 Harmonics

Further vertical lines for the harmonics can be superimposed. The starting frequency for the harmonics can be entered, i.e. the fundamental frequency. For example this can be set from the interpolated frequency of the first harmonic.

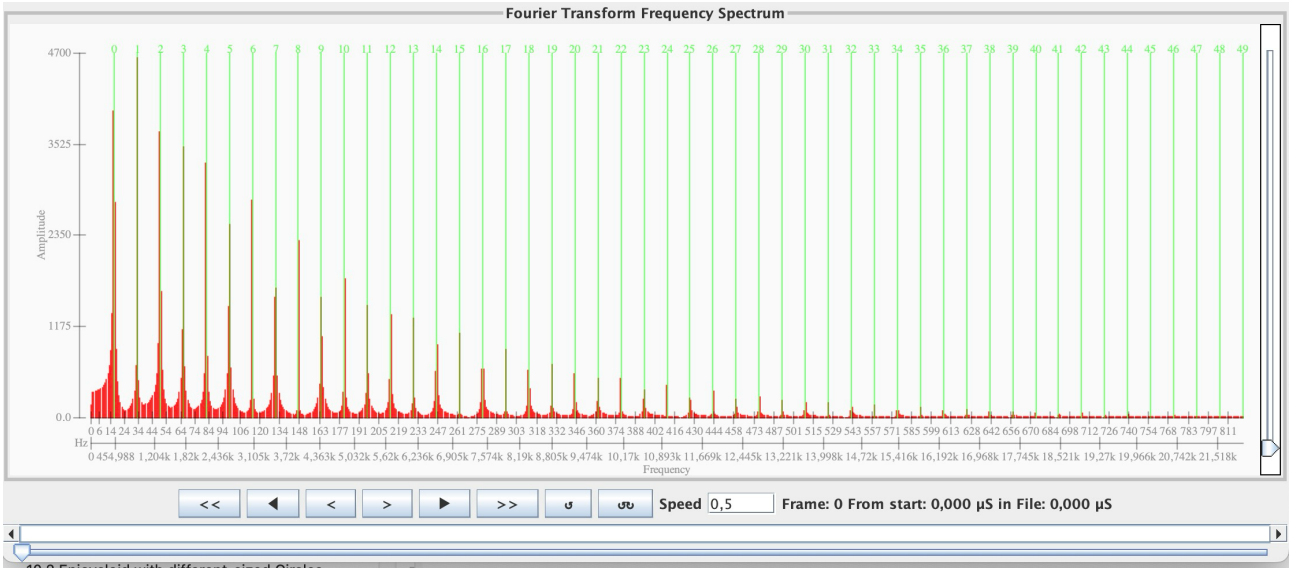


Figure 8-13 Spectrum Display with Harmonics

8.5.4 Notes

The notes of the musical scale can also be superimposed on the spectrum. The frequencies of the notes are determined from the entered frequency of A, e.g. 440Hz and the selected temperament.

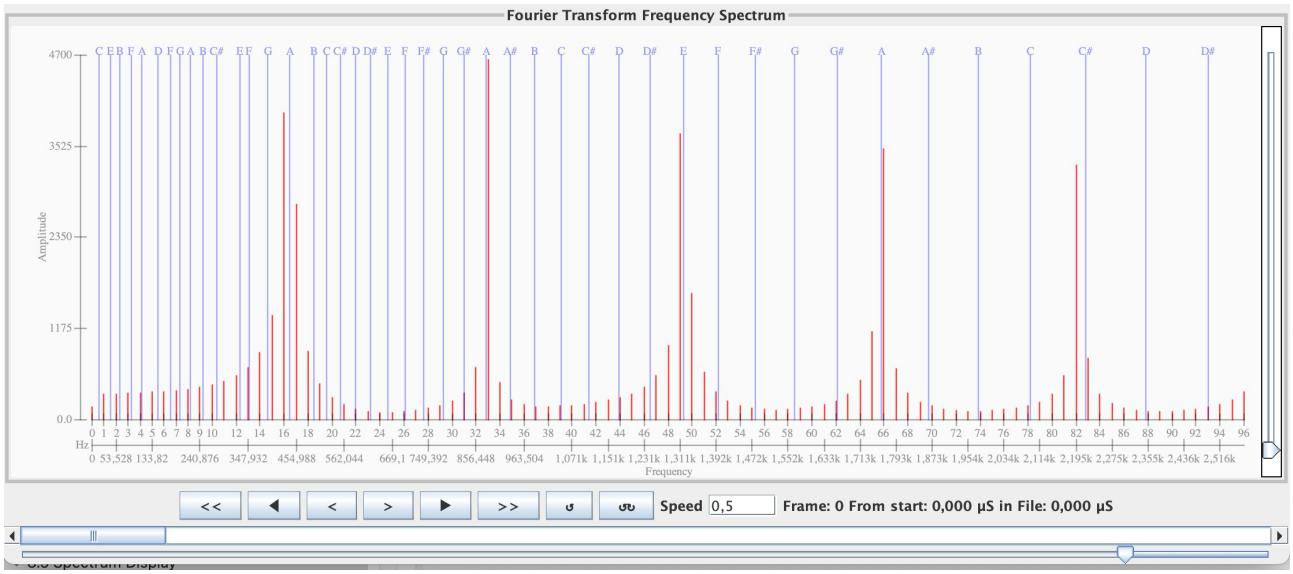
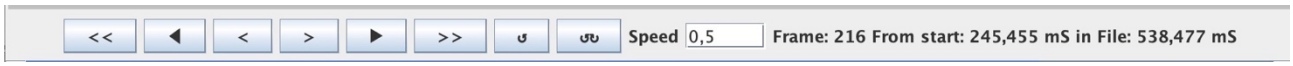


Figure 8-14 Spectrum Display with Notes

8.5.5 STFT Animation Controls

When a Short Time Fourier Transform has been done, the animation controls appear below the spectrum plot. These control how the frames are viewed and are as follows:



<<	Go to first frame (number 0)
◀	Play backwards. When clicked the frames are played from the previous position. The button shows a pause sign to pause the animation.
<	Step backwards by one frame
>	Step forwards by one frame
▶	Play forwards
>>	Got to the last frame
	The buttons that play the animation show a pause sign while running, and clicking this will pause the animation. The Go to and Step buttons will stop any running animation.
↺	Loop: play repeatedly from beginning to end
↺↻	Bounce: play forwards then backwards repeatedly
Speed	Sets the playback speed. 1.0 is real-time, i.e. if the frames cover one second it will play in one second. 0.5 will play half as fast, 2.0 will play twice as fast, etc.
Frame:	indicates the frame being currently shown. Note: this starts at 0 and ends at 1 less than the number of frames.
From Start:	shows the time of the current frame from the start of the animation.
In File:	shows the time of the current frame from the start of the file.

The best way to save the animation is to make a video of the screen while it is running.

Note: the peaks are shown in the interpolation section of the Analysis Parameters – see 8.6.9.

8.6 Analysis Parameters

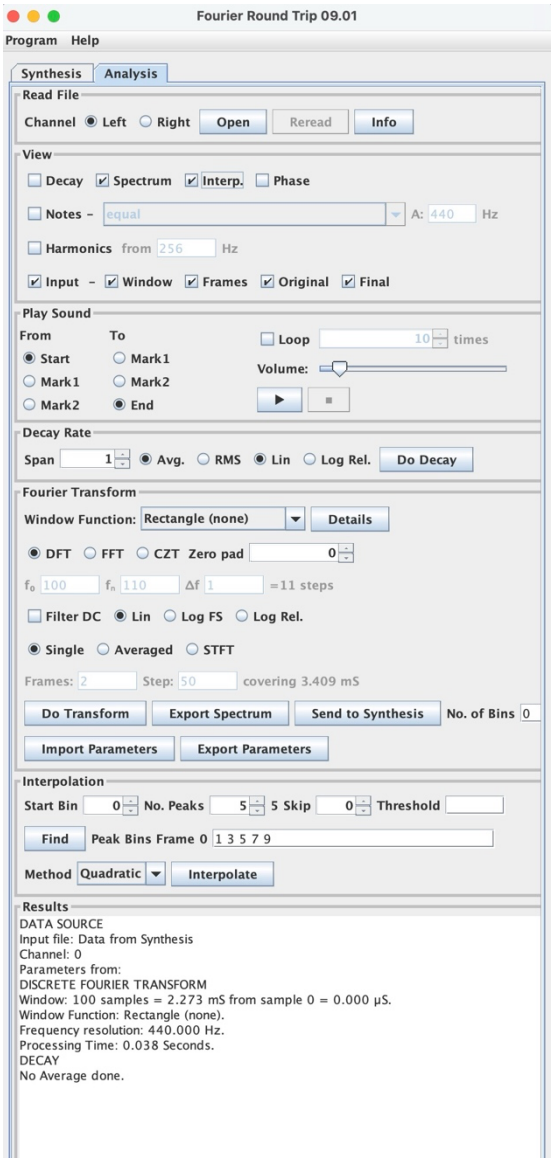
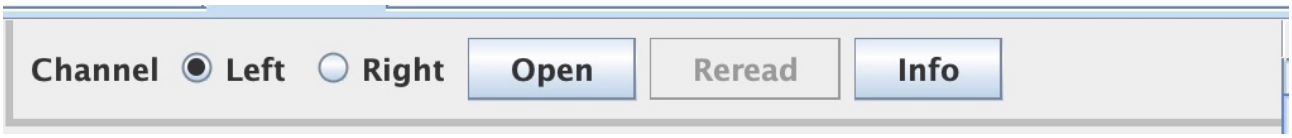


Figure 8-15 Analysis Parameters

8.6.1 Read File Section



This section controls the reading of wav files.
The channel to read can be selected before reading a new file or rereading the same file with a different channel. This is always available as it is not known in advance how many channels a file has. If "Right" is selected and only one channel is present, the single channel will be read.

Left

Read the left channel (channel 0).

Right

Read the right channel (channel 1).

Open

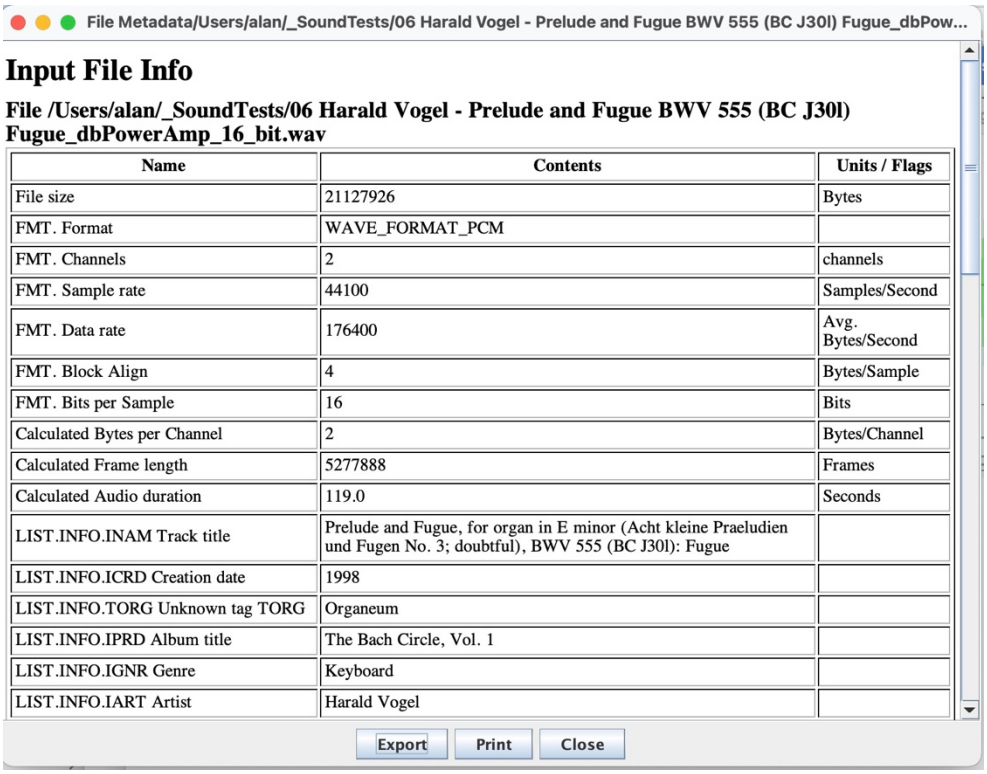
This opens a file chooser to select a new input file. For large files a progress monitor will display with the option to cancel the read.
If the read is cancelled or fails, the previous file remains open. After a successful read the analysis window is closed and re-created with the new file.

Reread

This reads the open file again, usually used after changing the channel.
Note: for making comparisons between channels, the analysis parameters can be exported before the reread and imported after rereading with the other channel.

Info

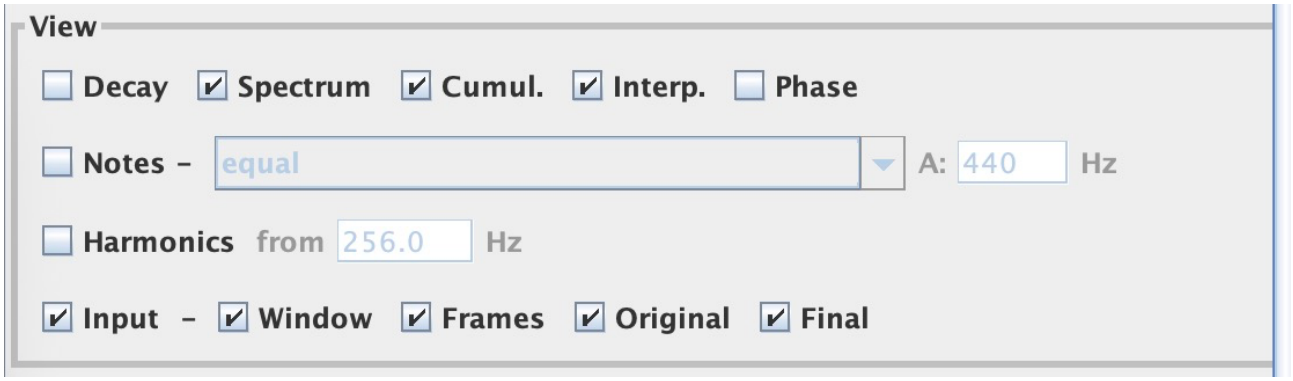
Brings up a window showing the information about the wave file being analysed. This shows the data derived from the formatting of the wav file and other meta-data contained in the file.
Entries are preceded by the internal tag name from which they are taken.
It attempts to show all embedded information but only shows the contents of those that are relevant; others have “skipped” in the contents.
There are buttons to export the content to an html file, print it and close the window.
Note that graphics will not be shown, as the built-in html interpreter does not support this, but exporting and displaying in the standard browser should show them correctly.
See [ID3 for details of the ID3 tags.



Name	Contents	Units / Flags
File size	21127926	Bytes
FMT. Format	WAVE_FORMAT_PCM	
FMT. Channels	2	channels
FMT. Sample rate	44100	Samples/Second
FMT. Data rate	176400	Avg. Bytes/Second
FMT. Block Align	4	Bytes/Sample
FMT. Bits per Sample	16	Bits
Calculated Bytes per Channel	2	Bytes/Channel
Calculated Frame length	5277888	Frames
Calculated Audio duration	119.0	Seconds
LIST.INFO.INAM Track title	Prelude and Fugue, for organ in E minor (Acht kleine Praeludien und Fugen No. 3; doubtful), BWV 555 (BC J30l): Fugue	
LIST.INFO.ICRD Creation date	1998	
LIST.INFO.TORG Unknown tag TORG	Organeum	
LIST.INFO.IPRD Album title	The Bach Circle, Vol. 1	
LIST.INFO.IGNR Genre	Keyboard	
LIST.INFO.IART Artist	Harald Vogel	

Figure 8-16 File Info Window

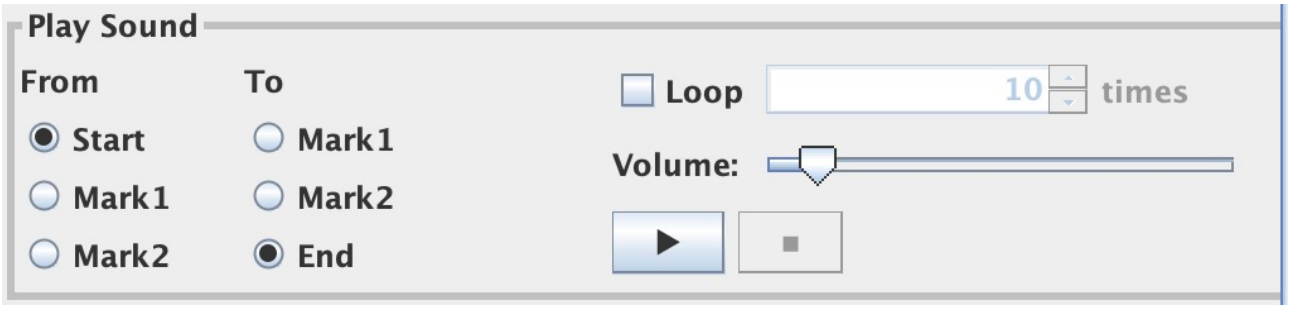
8.6.2 View Section



This section controls which panels and contents are shown in the main window.
Decay: the average and decay panel
Spectrum: the spectrum plot can be shown or omitted.
Cumul.: the cumulative spectrum can be shown or omitted.
Interp.: the interpolated spectrum frequencies can to shown or omitted.
Phase: the phase plot can be shown or omitted. Omitting it gives extra vertical space to the wave and spectrum plots.
Notes: the positions of musical notes can be shown as vertical blue lines in the spectrum (see Figure 8-14). The temperament can be selected from the drop-down list and the frequency of A can be set in Hz. The temperaments are taken from [BDO].
Note: note lines whose label would overlap the previous label are omitted.
Harmonics: the positions of the harmonics can be shown as vertical lines in the spectrum (see Figure 8-13). The frequency of the fundamental (harmonic number 0) can be set in Hz.
Input: the Input Wave panel with details:

Window: the outline of the window-function is shown.
Frames: the frames of an Average or STFT are shown with their starts marked as ticks above the plot and, if Window is checked, the window-function at each frame.
Original: the original wave is shown.
Final: the final wave after application of the window-function and the addition of zero padding is shown.

8.6.3 Play Sound Section



This section makes it possible to listen to all or parts of the sound file on the computer audio system in various ways.
From and To
The radio buttons in the From column (**Start**, **Mark1**, **Mark2**) determine from where the playback will start.
The radio buttons in the To column (**Mark1**, **Mark2**, **End**) determine where the playback will end.
The combinations are limited to those which are logically possible.

Loop and times

If this box is checked the selected part of the sound will be played repeatedly for the number of times given in the field. The duration of the window size between the marks is shown in the status bar at the bottom of the screen window, and the actual playing time of the loop is given by multiplying this by the number of loops entered.

Volume

Set the desired playback volume.

Note: the volume is also affected by the volume setting on the computer itself and the connected amplifier and speakers.

Note: some audio formats do not support a volume control. In this case the slider is disabled and greyed out, and the wave plays with the equivalent of full volume on the slider – reduce the computer volume!

Play/Pause ►⏸

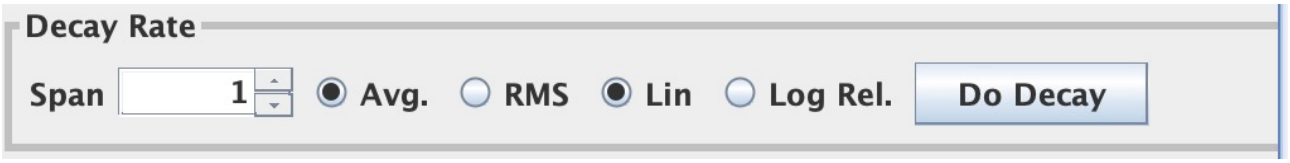
This button alternates between Play and Pause. When stopped, the play button starts playing all or part of the sound as selected above. When playing, the button pauses the sound. When paused, the button continues playing the sound.

When paused, the cursor marking the current playing position can be moved and the play button will continue from there. Note 1: if the cursor is moved outside the chosen From and To positions, the play button will effectively stop as if the playback had ended. Note 2: If the cursor is moved, the loop will not continue, but playing will stop at the chosen end point.

Stop■

The stop button stops the currently playing sound.

8.6.4 Decay Rate Section



A moving average of the input wave between the two marks is calculated and displayed.
The average is marked as "stale" when any of the following parameters are changed or the marks are moved.

Span

Set the number of samples over which to average. Each averaged sample is derived from the samples on either side so that the averaged wave does not have an offset on the time axis. The span must therefore be odd. The marks must be at least half the span from the ends of the file.

The span is shown on the Average and Decay graphic near the top of the marks in the same colour as the mark.

Avg. or RMS

Select whether the average is taken with the arithmetic mean or the Root Mean Square.

Lin

Check to use a linear scale with the sample values.

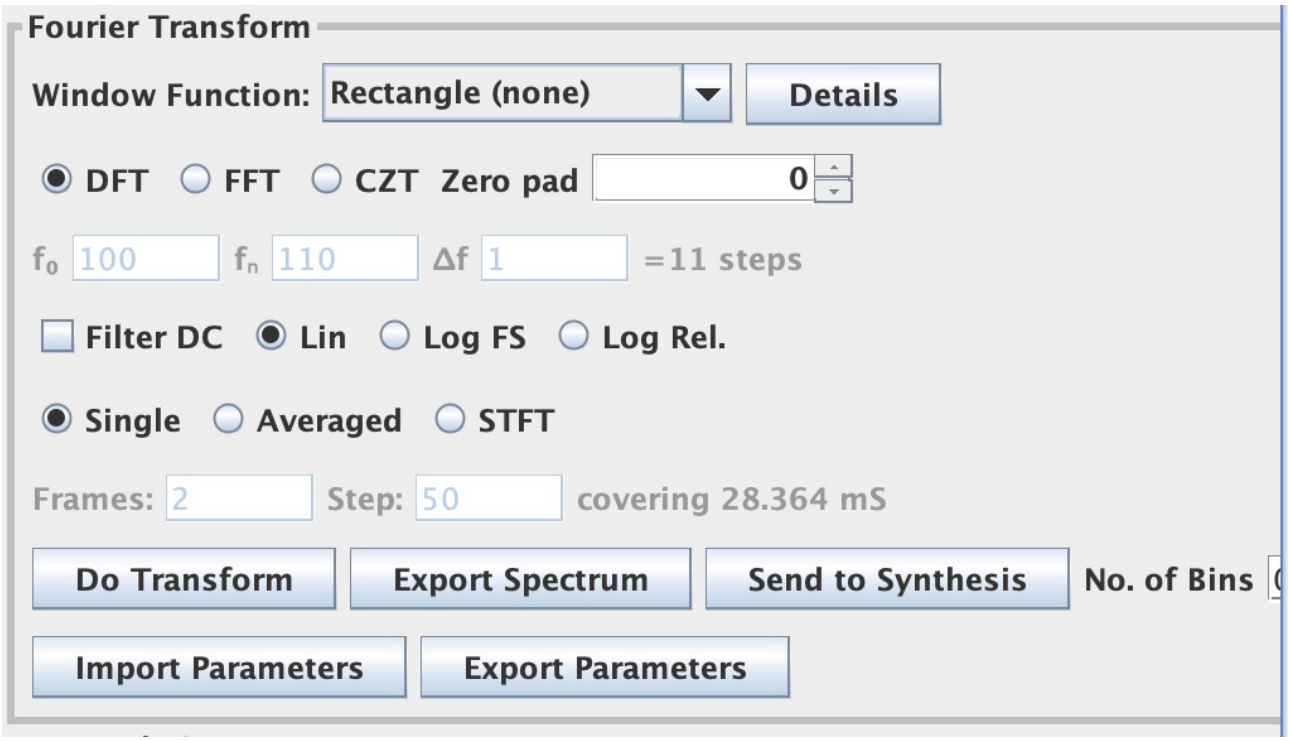
Log Rel.

Check to use a logarithmic scale in decibels relative to the maximum absolute value of the input.

Do Decay Button

Click the button to perform the decay rate calculation. A progress monitor is shown for each stage, averaging, decibel conversion and regression.

8.6.5 Fourier Transform Section



This section determines how the selected segment of the wave is to be modified before analysis.

Window-Function

The drop-down list selects the window-function applied to the sample window defined by the marks. Some functions have an extra parameter and a field for this will appear beside the function if required. See Appendix 18 for details.

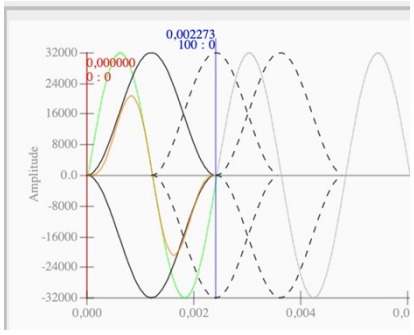


Figure 8-17 Example of a Hamming Window with 3 frames

Details

The Details button brings up a window with a plot of the window-function, its Fourier transform and some of its parameters – see section 8.7.

DFT, FFT and CZT

Select whether a Discrete Fourier Transform or a Fast Fourier Transform, or a Chirp Z-Transform is to be used (see section 3.13).

Note: for FFT the window size must be a power of 2. If it is not, the effective window is increased and filled with zeros at the start and end.

Zero pad

Enter the number of zero samples to add to the end of the sample window. This is shown on the wave plot by an extension of the final wave along the x-axis and a third mark (see 8.2.2).

CZT Parameters (only visible when CZT is selected)

f_0

Enter the starting frequency of the CZT.

f_n

Enter the end frequency of the CZT

Δf

Enter the frequency increment for the CZT

The resulting number of steps is shown.

Note: it is usually better to enter the end frequency first, as the end frequency must be greater than the starting frequency.

Filter DC

If this is checked, the DC (direct current or constant bias) will be filtered out of the window before doing the Fourier Transform. The average of the wave over the window is subtracted from all the samples within the window.

Lin, Log FS, Log Rel.

These radio buttons determine how the amplitude (y-axis) of the resulting spectrum is scaled and how the vertical slider on the right of the spectrum acts.

Lin - Linear scaling. The slider magnifies the spectrum, and higher amplitudes will disappear off the top of the plot.

Log FS – Logarithmic Full-Scale scaling in dB relative to the maximum possible value of the sample size. Moving the slider up increases the lower limit shown.

Log Rel. – Logarithmic scaling in dB relative to the highest amplitude in the spectrum. Moving the slider up increases the lower limit shown.

The following radio buttons select the type of transform to be performed.

Single

When Single is selected a single Fourier Transform of the sample window is performed using the type of transform DFT, FFT or CZT selected above.

Averaged

When Averaged is selected Fourier Transforms are performed for the window multiple times for the given number of frames separated by the step (see below) and the results are averaged to give the spectrum and phase plots.

Note: the number of frames, the amount of overlap and the shape of the wave can produce unexpected results e.g., a single sine wave with an even number of overlapping rectangular windows with 50% overlap will give a zero result because they cancel each other when averaged.

STFT

When STFT (Short Time Fourier Transform) is selected, Fourier Transforms (DFT, FFT or CZT) are performed for the window multiple times for the given number of frames advanced by the step and the resulting spectra are displayed in an animation. The phase spectrum is shown for the first frame only.

STFT Parameters (only visible when Averaged or STFT is selected)

Frames

Select the number of frames to be transformed for Averaged or STFT. The frames are shown with their window-function in the Input Wave graphic. If Single is selected above only one frame is shown.

Step

This determines the distance in samples between the start of one frame and the start of the next. This should be chosen based on the window size (shown in the status bar

below the Input Wave), e.g., 20% of the window size to give 80% overlap. The corresponding percent overlap will be shown in the results (see 8.6.10).

covering

The effective time covered by the combination of Frames and Step is shown so that it can be related to the Input Wave time scale.

Note: if the frames go beyond the end of the file an explanatory error message is shown, and the number of frames is set to the maximum value that will fit. Zero padding or expansion for the Fast Fourier Transform can extend beyond the end of the file.

8.6.6 Fourier Transform – Do Transform Button

Do Transform

This starts the Fourier transform analysis as set up with the marks in the Input Wave and the parameters above. The progress is shown in the results section.

A single transform is performed on one processor with one progress bar.

For Averaged and STFT the work is divided among the available hardware processors[6] with a progress bar for each. Each processor is started on a frame and when completed, it starts on the next available frame. Since they work asynchronously the frames will not be allocated to processors in any particular order.

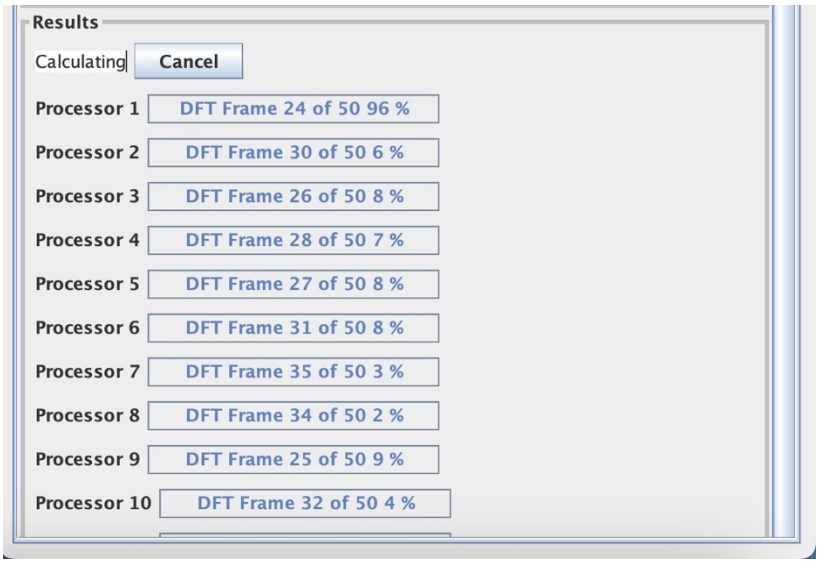


Figure 8-18 Analysis Progress

The transform is performed in up to three parts and the progress bars display the frame on which the processor is working "of" the total number of frames and the % progress of that frame, preceded by the part:

Part 1, the actual transform, shows either "DFT" or "FFT".

Part 2a, deriving the amplitudes and phases from the complex results of part 1, shows "Part 2a".

Part 2b, the logarithmic scaling if "Log FS" or "Log Rel." was selected, show "Part 2b".

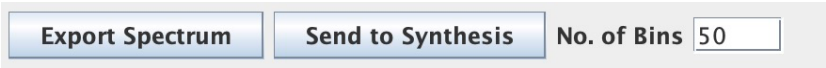
Some calculations may happen so fast that no progress bar appears. When all calculations are complete, the result is displayed – see 8.6.10.

Cancel

The cancel button appears in the results window while the transform is running and stops the analysis.

[6] It leaves two processors for other work, so in this example 12 of the available 14 processors are used.

8.6.7 Export Spectrum and Send to Synthesis



These buttons respectively either:

- write the frequency spectrum to a CSV file as a table of frequency, amplitude and phase, which can then be read in Excel or imported into the Synthesis function
- or send the spectrum directly to the Synthesis.

See section 7.3.6 for the format – the description is the title of the Analysis Window (e.g. the name of the file opened or "Analysis from Synthesis") and the timestamp; the frequency is the frequency of the first bin; the Noise is set to false, i.e. none.

The number of frequency bins exported or sent is limited by the "No. of Bins" field – see below.
For STFT the first frame is sent.
When sending from a CZT to synthesis or importing a CZT exported file, the frequency in the synthesis is set to that starting frequency f_0, so will have to be set manually to the actual fundamental frequency of the original spectrum.

No. of Bins

Use this to limit the number of frequencies, i.e. frequency bins, to export or send to the synthesis. An entry of 0 will send all of them. The maximum is half of the sample rate.

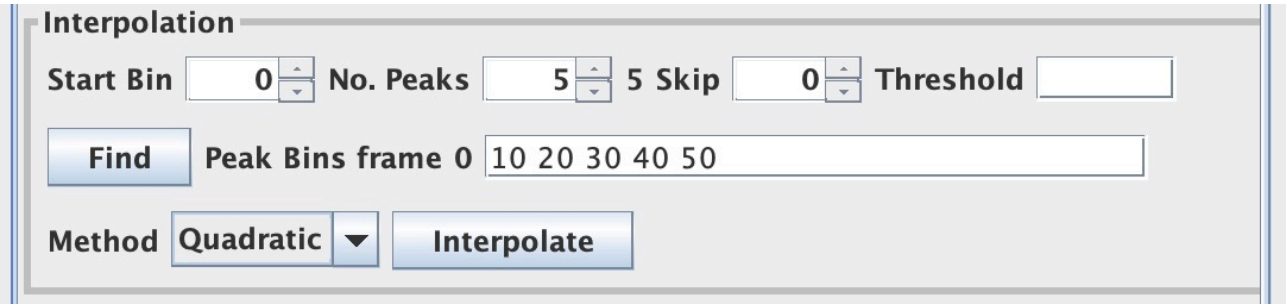
8.6.8 Import and Export Parameters



This function enables analysis results to be reproduced.
All the parameters set in the Analysis tab as well as those set in the Analysis window (zoom and mark synchronisation, STFT playback speed) can be exported to a properties file and imported again.
The import is in two separate parts, one for the transform parameters and one for the interpolation parameters.
The import directory will start at the one used previously, or the last used export directory, or the user's home directory in that order.
The export directory will start at the one used previously or the last used import directory, or the user's home directory in that order.
If either of the imported marks are beyond the end of the current file, they will be set to the end of the wave.
The path of the sound file for which the parameters are saved is in the file, but this is not used. If used with a different file, the results may not all be applicable.
The file is in a human-readable text format.
Note: the transform parameters must be imported after reading the audio file and before starting a transform. The Interpolation parameters must be imported after a transform has been done and before starting an interpolation.

8.6.9 Interpolation Section

This section is used to find the peaks in the spectrum and perform interpolations to find a better



approximation to the peak frequency and amplitude.
The search for peaks uses a simple algorithm of following the spectrum bins' amplitudes as they rise until the next bin's amplitude is less than that of the previous bin. The results can be manually edited if the result is not satisfactory or if it is desired to ignore specific peaks. A threshold can be set to exclude unimportant peaks or those generated in the noise level.
The peaks are shown as crosses along the top of the spectrum plot.

Start Bin

Set the bin from which to start searching for peaks. This could be used for example, to exclude the DC in bin 0 or to concentrate on a frequency band of particular interest.

No. Peaks

Select the maximum number of peaks to find.
The number of peaks found appears to the right of this field, as it may be less than the desired number. This will also change if peaks are manually added or removed – see below.

Skip

If the peaks are not clearly defined, i.e. there are similar amplitudes adjacent to each other, a number of bins can be skipped before the next peak is sought.

Threshold

Enter the minimum amplitude value for a peak. Any peaks below this value will be omitted.
Note: for logarithmic scales the value must be given as negative decibels. See the Amplitude scale on the y-axis of the spectrum.

Find button

Click the Find button to find the peaks. The result is shown in the Peak Bins field and the number found is shown to the right of the No. Peaks field. The peaks are marked by crosses in the spectrum plot.

Peak Bins frame n

This field shows the bin numbers of the peaks found. The field can be edited to insert or remove bin numbers. The bin numbers will be sorted and excess spaces removed when the field loses focus.
The frame *n* is determined by the STFT animation – see below.

Method

Select the interpolation method.
See Appendix 16 for details of the methods.

Interpolate Button

Click this button to perform the interpolation.
The results are shown in the spectrum plot with labels giving the interpolated frequency (in Hz) and amplitude (in the scale of the spectrum).

Note for STFT

With the Short Time Fourier Transform, the peaks and interpolation are performed for each frame of the transform. The Peak Bins and the actual number of peaks is shown for the frame currently being viewed in the STFT animation – see 8.5.5.
Note: see Appendix 17 on using this to find inharmonicity.

8.6.10 Results Summary

At the bottom of the window a summary of the result of the analysis is shown as follows (Note: the examples do not necessarily come from the same input.)

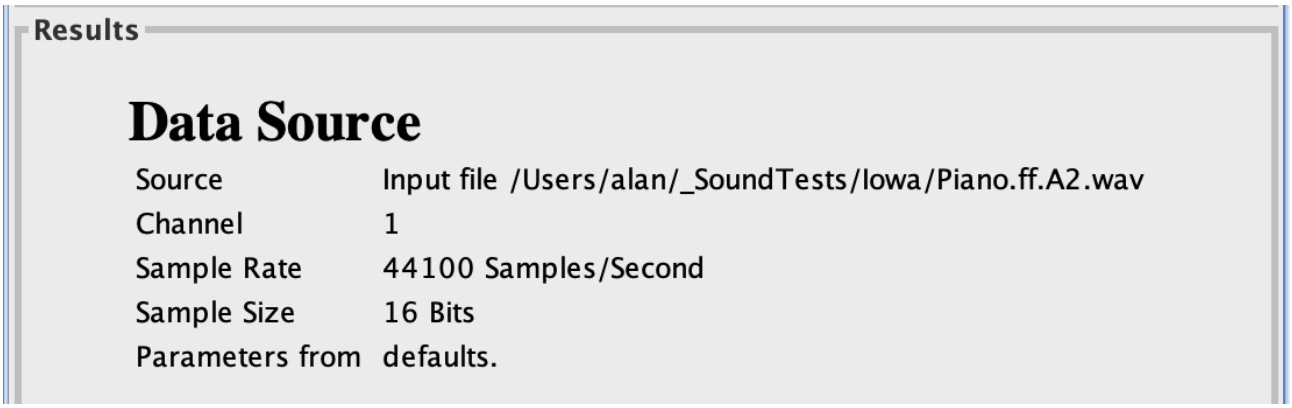


Figure 8-19 Results: Data Source

Data Source:

- Input file path
- Channel (0 = left, 1 = right)
- Sample rate
- Sample size
- Parameters from: the path of the file from which parameters were last imported.

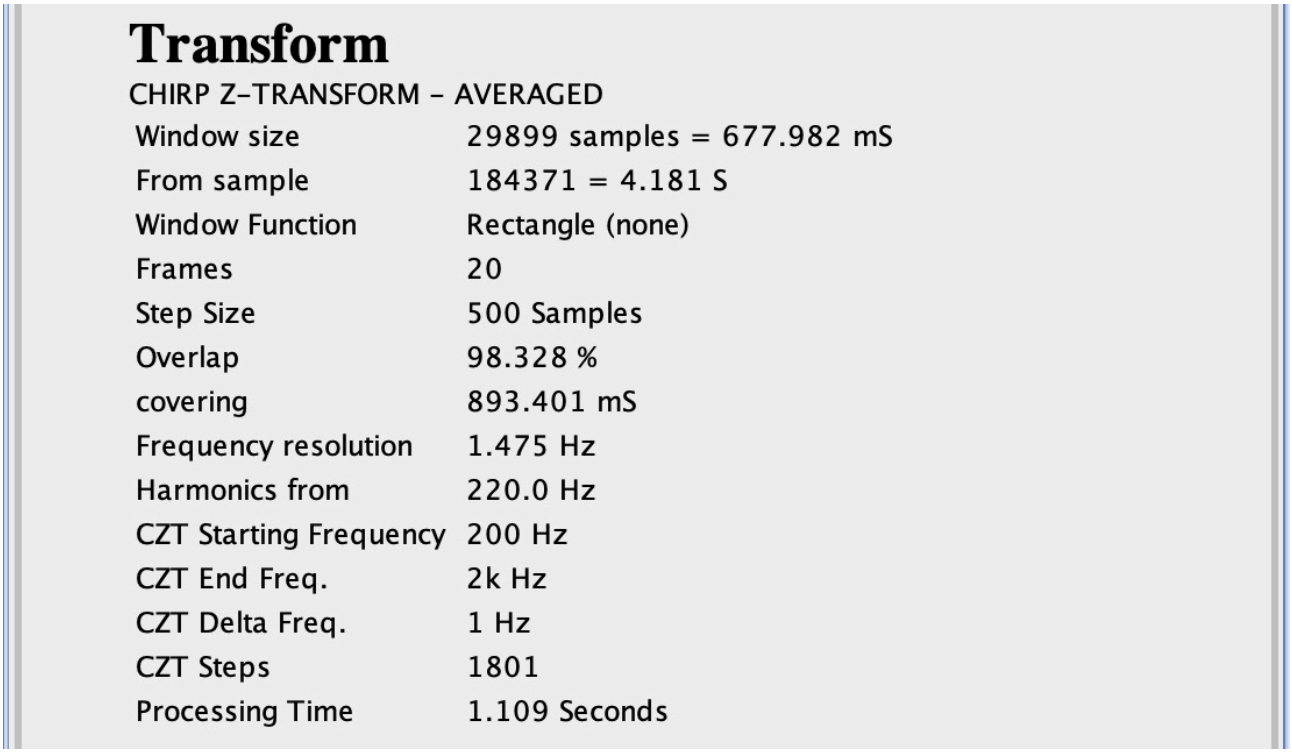


Figure 8-20 Results: Transform

Transform

Type of Transform e.g. DISCRETE/FAST/CHIRP Z FOURIER TRANSFORM and SINGLE/AVERAGED/SHORT TERM FT ANIMATION

- The analysis window size in samples (without padding or expansion) and its duration
- The position of the first sample (mark1) and its time in the sound file

- The size of the window including zero padding (if any) in samples and in milliseconds
- The size of the window after expansion to a power of 2 for Fast Fourier Transform in samples and in milliseconds
- The windowing function and its parameter(s) (if any)
  For Averaging of STFT:
  - The number of frames
  - The step size in samples
  - The percentage overlap
  - The time covered
- The frequency resolution, i.e. the difference in frequency between the individual spectral lines
- If musical notes were enabled, the temperament and frequency of A.
- If harmonics were enabled, the fundamental frequency.
  For CZT:
  - The starting frequency
  - The end frequency
  - The delta frequency
  - The number of steps
- The processing time taken to process the analysis.

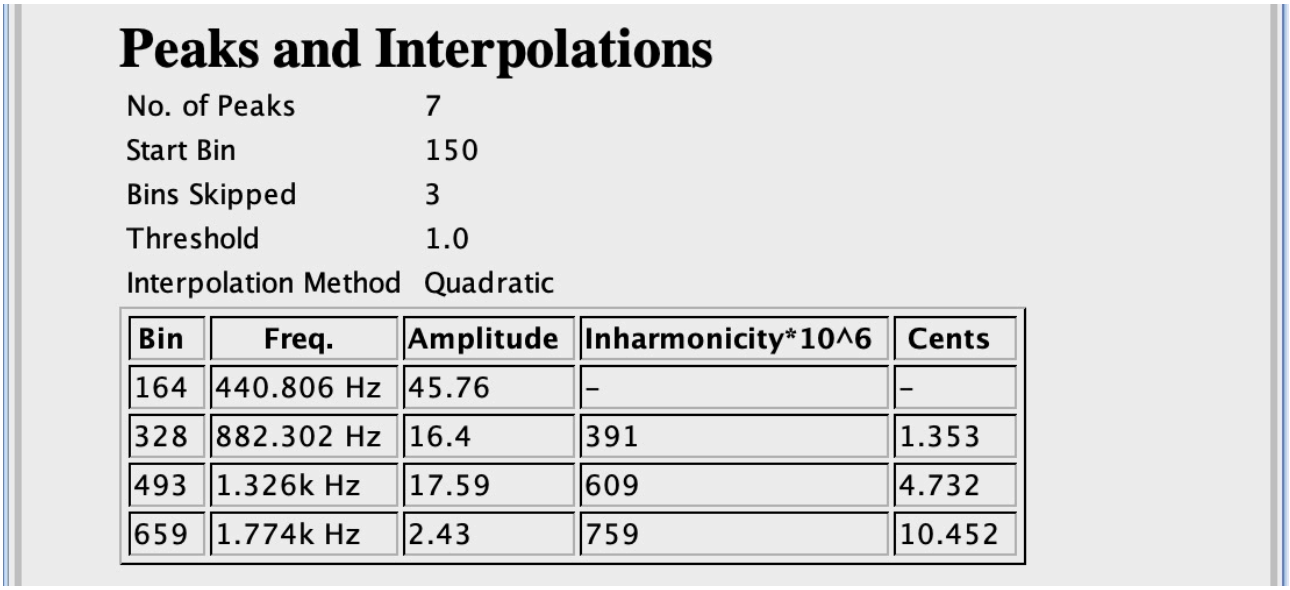


Figure 8-21 Results: Peaks and Interpolation

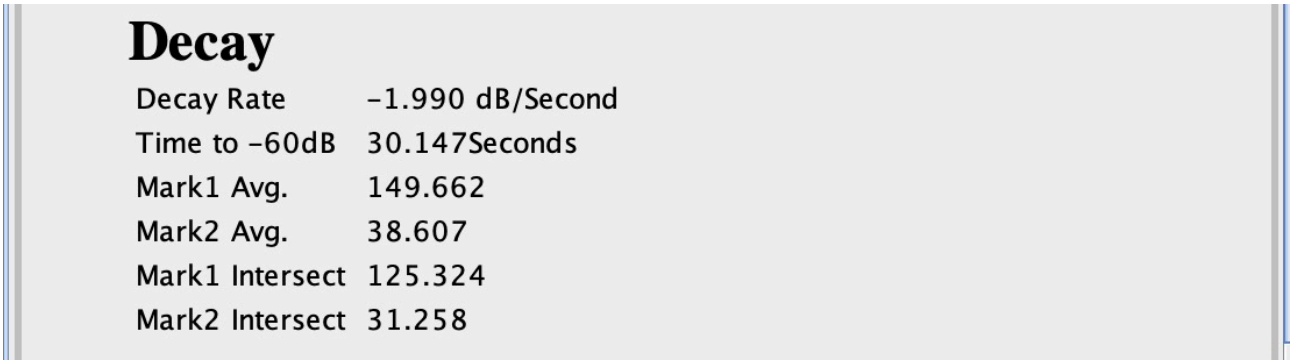
Peaks and Interpolations

- The number of peaks
- The starting bin of the peaks
- The number of bins skipped after a peak
- The amplitude threshold for detecting a peak
- The interpolation method

A list of the interpolated peaks with:

- bin number
- frequency
- amplitude
- the inharmonicity coefficient
- the inharmonicity in cents

Note: see Appendix 17 for details of inharmonicity.



Decay

Decay Rate	-1.990 dB/Second
Time to -60dB	30.147Seconds
Mark1 Avg.	149.662
Mark2 Avg.	38.607
Mark1 Intersect	125.324
Mark2 Intersect	31.258

Figure 8-22 Results: Decay

Decay

- The decay rate in decibels per second calculated from the linear interpolation of the averaged wave.
- The number of seconds to reduce the amplitude by 60dB (commonly used as a measure of the reverberation time of rooms such as concert halls). Note: this is negative if the wave is increasing in amplitude.
- The average value at mark 1 and mark 2 over the span.
- The interpolated values and the intersects of the regression line at mark 1 and mark 2.

The results are in html with tables and can be copied to the clipboard for use in other programs such as text processors or spreadsheets.
Values are replaced by “Cancelled or not started” if the analysis was not complete.

8.7 Fourier Analysis-Window Function Details

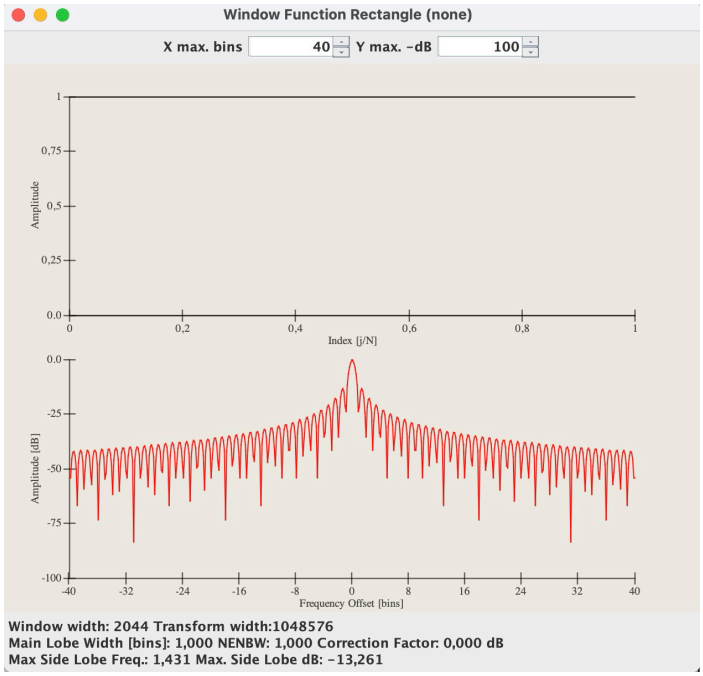


Figure 8-23 Window Function Details

This window is brought up by the Details button for the analysis window – see section 8.6.5.
This shows the function name in the title bar and has fields for the display parameters as follows:

X max. bins: The number of frequency bins to show on each side of the x-axis. A high value of this can take a long time to run and may cause the program to run out of memory and crash.
Y max. -dB: The lower limit on the y-axis in dB below which values are cut off.

The window can be resized, but aliasing will occur as the samples of the transform are mapped to the pixels of the display. If the values of “X max. bins” or “Y max -dB” are changed, the display will be recalculated.
Display windows are not closed automatically so the different functions and settings can be compared.
The parameters shown are as follows:
Window width: The number of samples in the window -function, set at 4 times the plot width in pixels.
Transform width: The number of frequency bins in the transform. This is the next power of 2 above 10 times the window width.
Main Lobe Width [bins]: The width of the main lobe in frequency bins. This is half of the width seen on the graph, i.e., only one side.
NENBW, the Normalised Equivalent Noise Bandwidth: The width of a rectangular bin that would contain the equivalent amount of noise.
Correction Factor: The correction factor in dB resulting from the NENBW which should be applied to the spectrum values.
Max. **Side Lobe Freq.**: The normalised frequency of the highest side lobe (not the first side lobe in all windows).
Max. Side Lobe dB: The value in decibels of the highest side lobe.
Note that last two values are not always correct depending on the choice of “X max. bins” and the aliasing from the window width.

Incidentally, the Rialto Bridge (see Frontispiece 1) has a fall-off rate of about 11.5 dB/Octave, taking the distance between the arches to be an octave!

9 Experiments in Synthesis

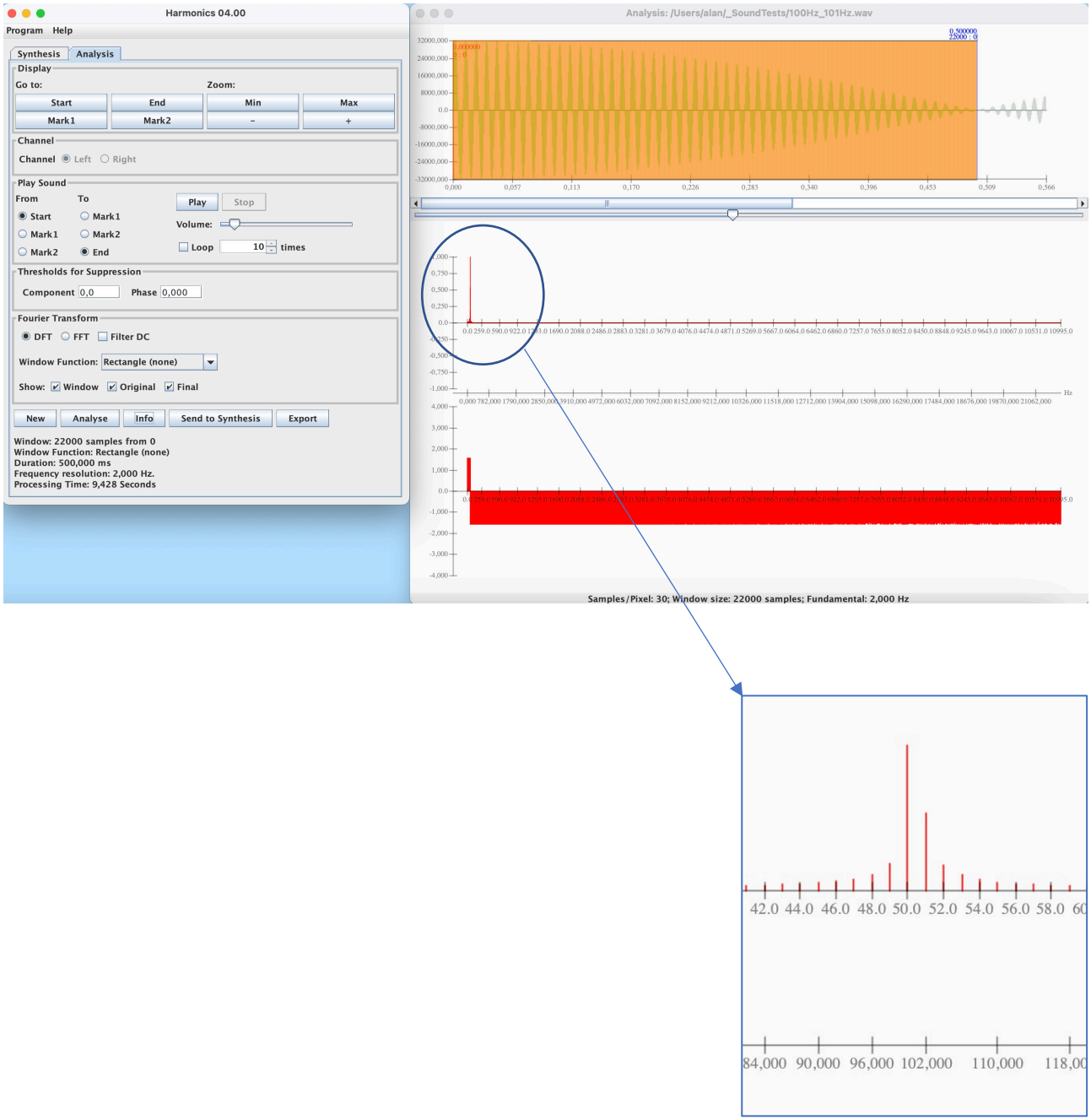
9.1 Overtones

See section 3.4.
Select 10 harmonics.
Observe that with a progression of 1/n (for example a plucked string) the wave tends to a sawtooth shape, but with $1/n^2$ (struck string) it is smoother. Note the difference in the timbre.

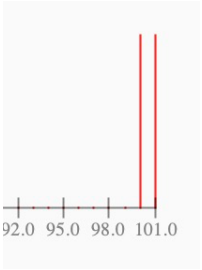
10 Experiments in Analysis

10.1 Two Adjacent Frequencies

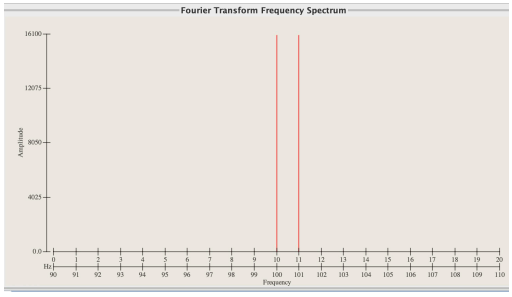
Synthesise or read in a signal with sine waves at 100Hz and 101Hz with a sampling rate of 44000 samples/second.
Analyse using half a second window (50 cycles of the 100Hz wave) the frequency resolution is 2Hz.
The two spectral lines 1Hz apart cannot be resolved.



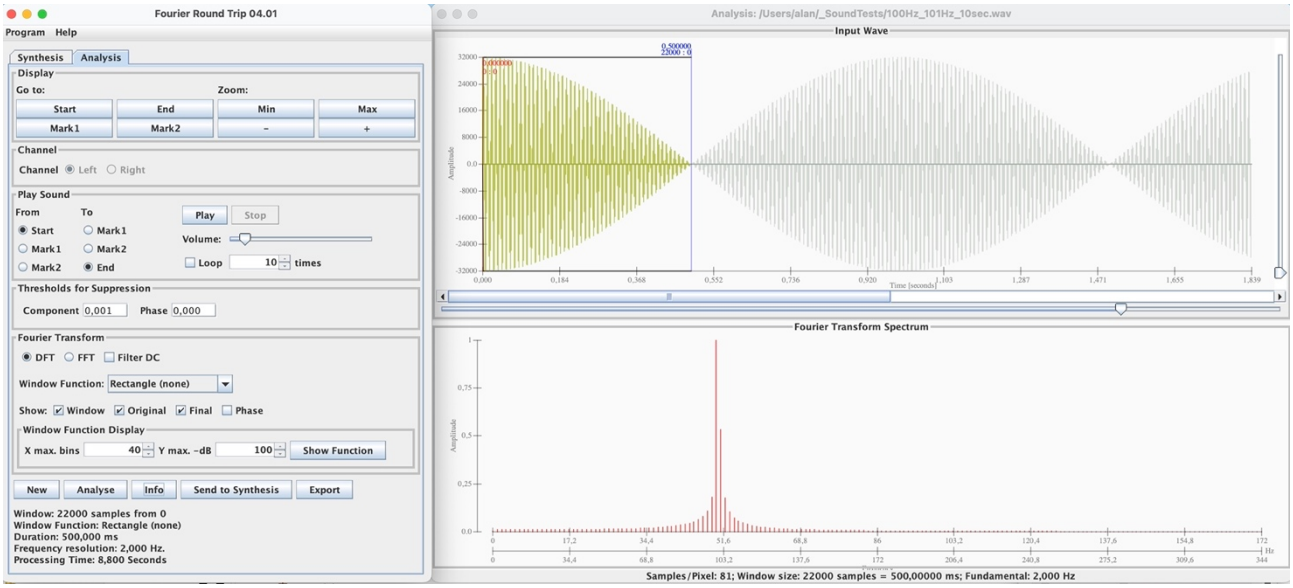
The 100 Hz signal is there because it falls on a frequency division, but the 101Hz signal is distributed over the whole spectrum, mainly in the 102Hz bin and more above 100Hz than below.
If we now take a higher sample rate of 131072 samples/second (chosen to be a power of 2 for an optimal FFT), with the window over the entire 1 second, we obtain a resolution of 1Hz and can see both frequencies:



A CZT of 90 – 100Hz in 1Hz steps is even clearer:

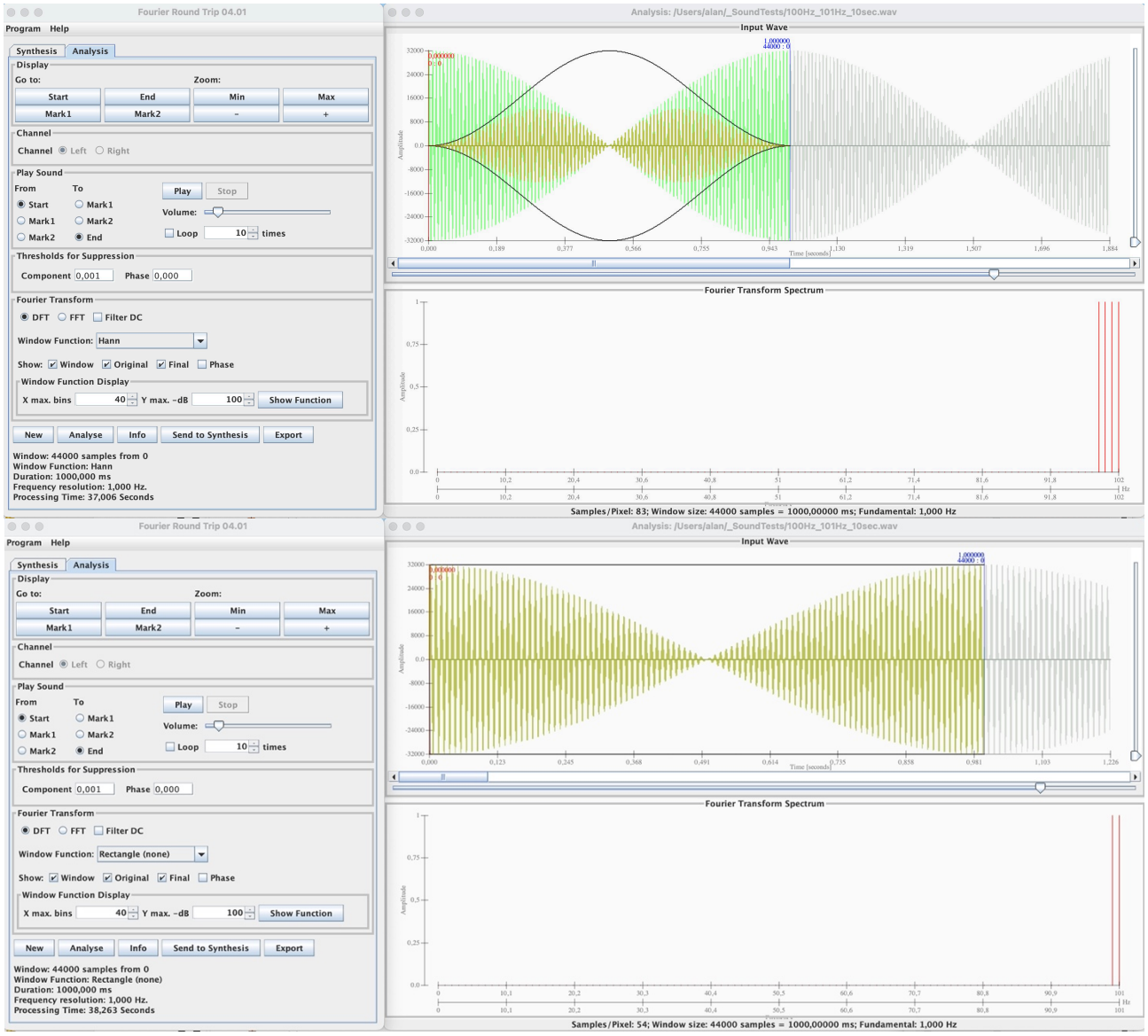


We can also see the effect of window size. Using a 10 second file at the lower sampling rate, taking a window over 0.5 seconds does not fully resolve both frequencies (see [Smith] p. 180ff.),

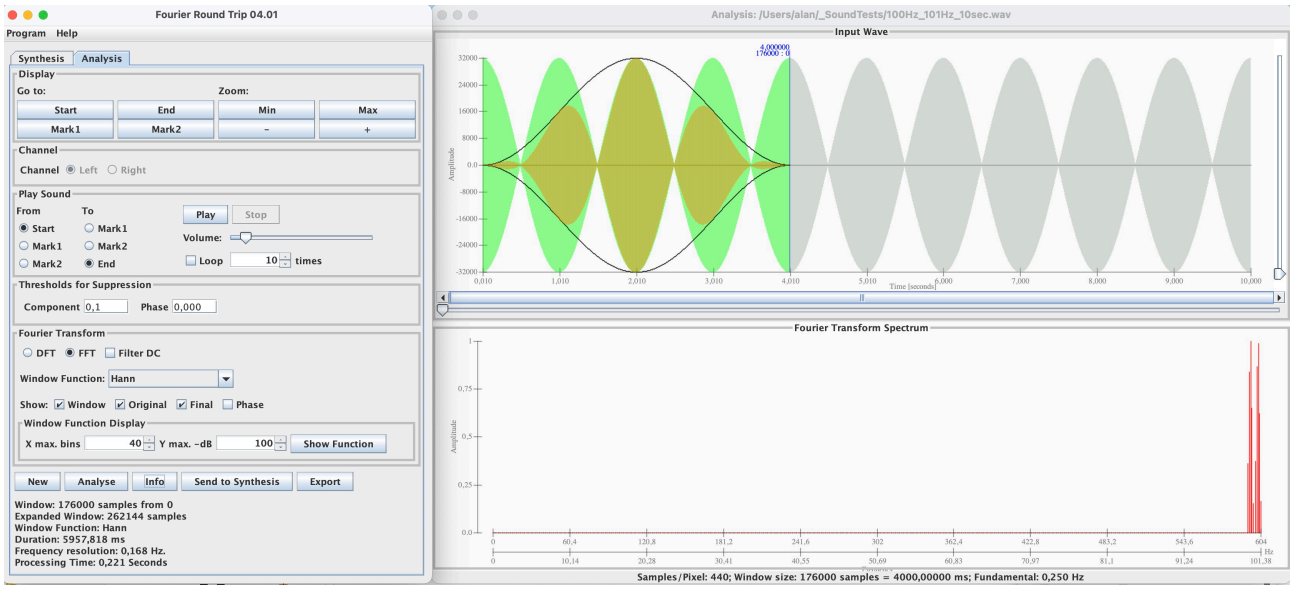


but using a window of over 1 second, to resolve the beat frequency of 1Hz, does resolve the two frequencies at the lower sampling rate.

A Hann window needs 4 seconds to resolve these frequencies. With 1 second:



and 4 seconds:



10.2 Sidelobes

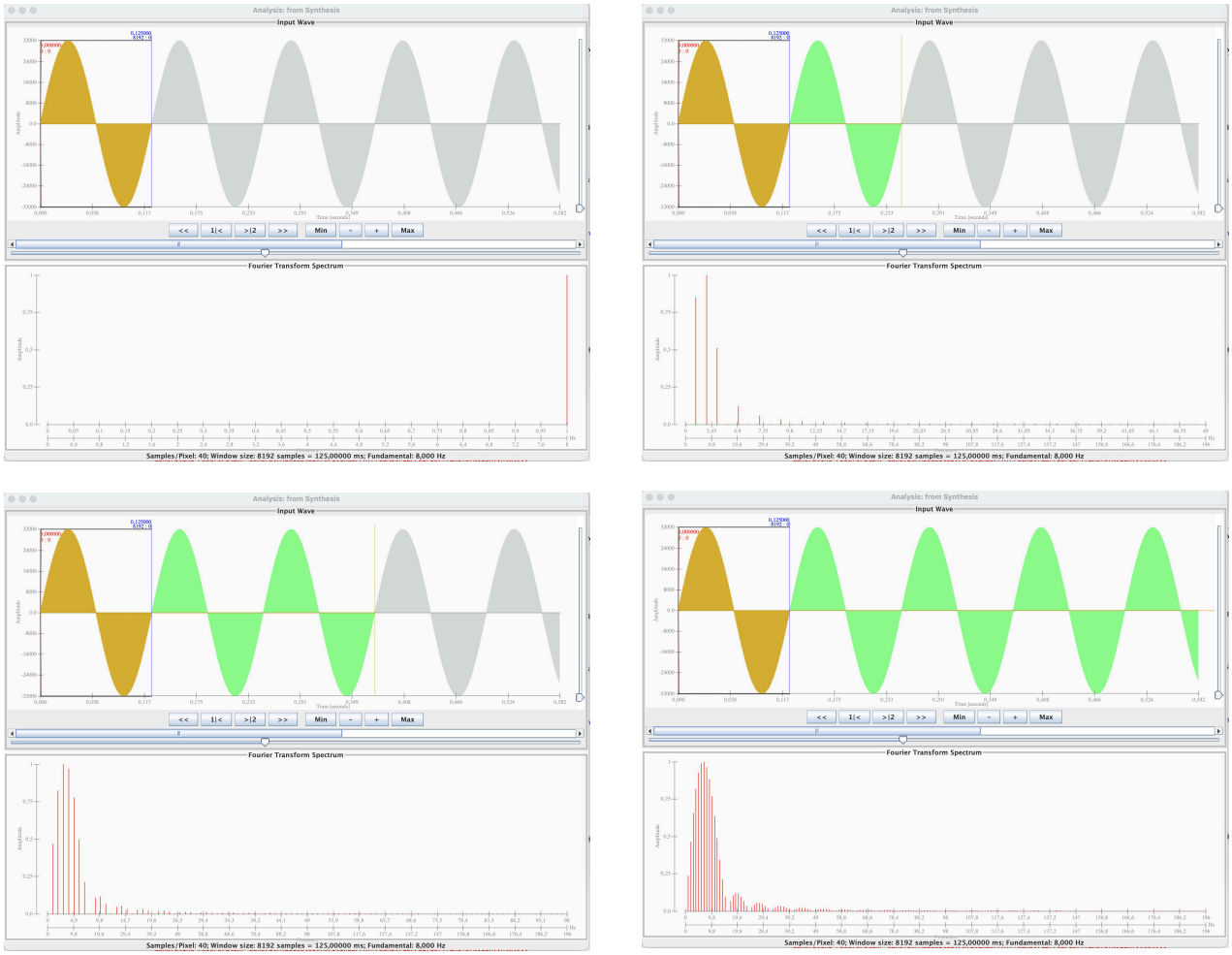
From https://www.akustik-messen.de/index.php/en/background-information/spectral-analysis?view=article&id=132:fft-analysis&catid=15:background-information

The higher the attenuation from the sidelobes caused by the leakage effect, the lower the frequency resolution. The main peak widens. If THD measurements are performed and you cannot use the certain frequencies mentioned above, the best results are achieved with the Rife-Vincent or Blackman windows. However, you get better results if block periodic frequencies are used.

The effect can be demonstrated easily. Set the generator to sine at 1kHz and the analyzer to FFT mode and block size of 1024. The sample rate should be set to 44.1kHz. With the rectangular window set to on, you will monitor sidelobes. If you vary the frequency, you will notice, that at certain frequencies no sidelobes occur. If you use a different window function, the sidelobes are attenuated independently from the input frequency.

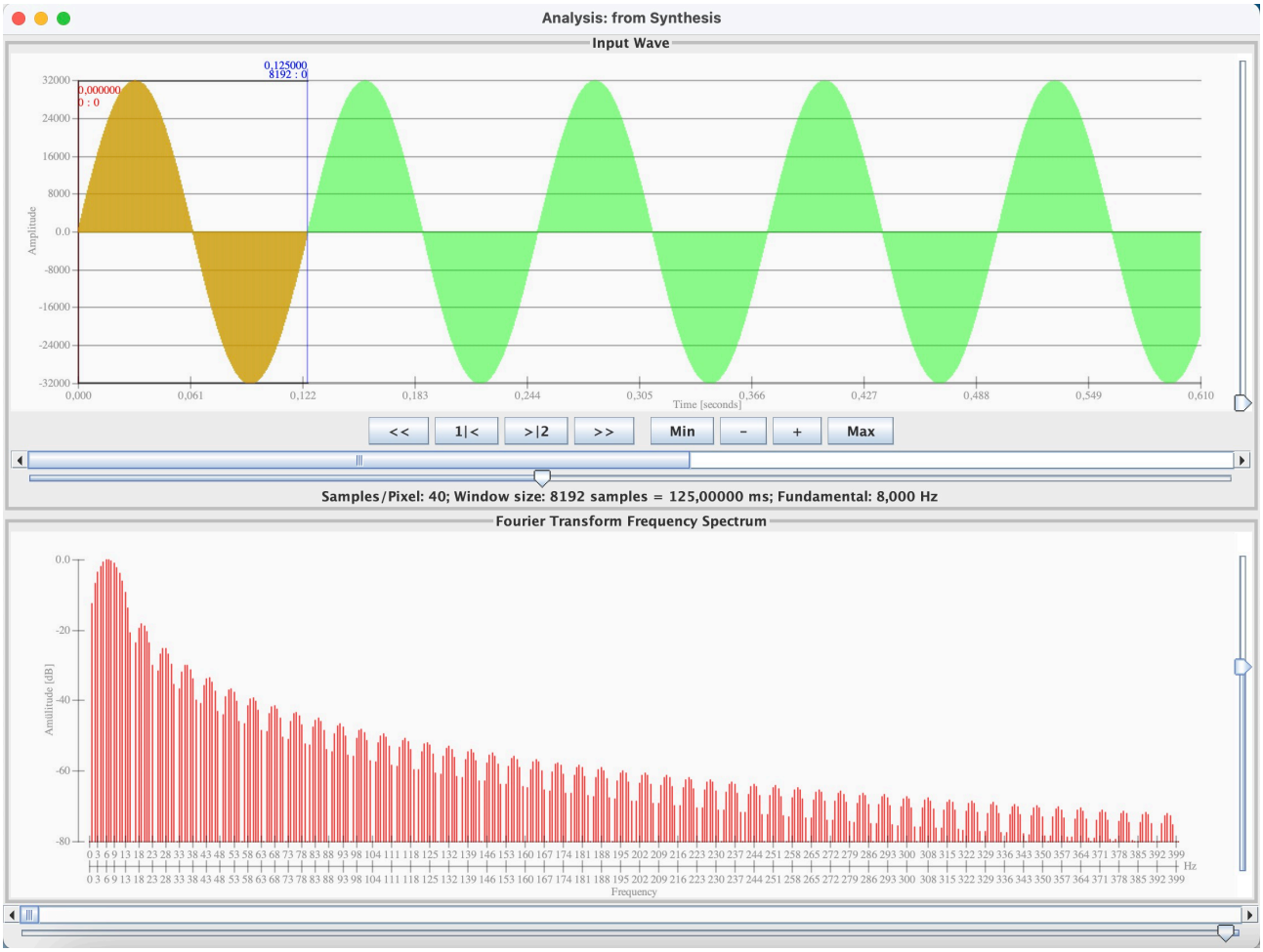
10.3 Zero Padding

Use a sine wave at 65536 samples/s and 8Hz to give an optimal FFT.

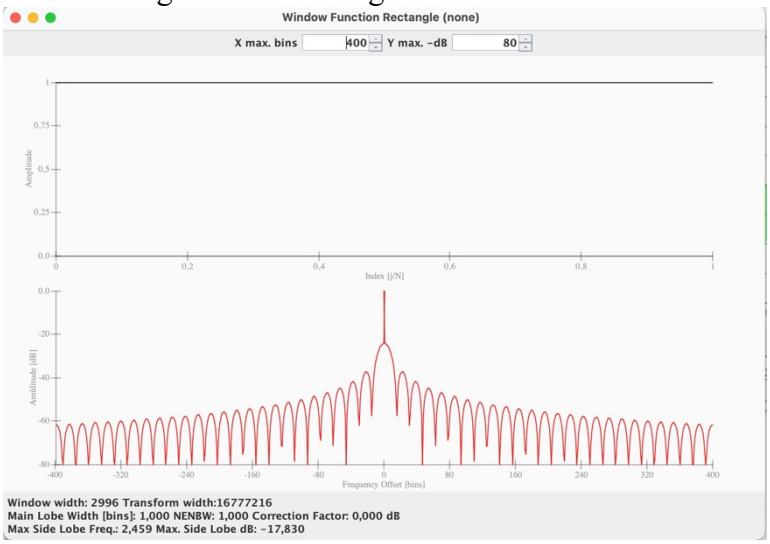


The first plot is with a window of 1 cycle = 8192 samples. Adding 8192, 16384, 32768 zeros shows better approximations to the continuous Fourier transform of the rectangular window

With logarithmic scale:



we can recognise the Rectangular window -function:



10.4 Sampling Rate

Using a sampling rate and a frequency that are powers of 2, e.g., 65536 samples/s and 8Hz.
TBD

10.5 Overlapping Analysis Windows

An even number of windows with 50% overlap averages to zero.
TBD

10.6 Noise Reduction

Use Synthesis to create a sawtooth wave (All harmonics) of say 3 harmonics and add noise with a peak of, for example, 0.7.
Send it to Analysis.
In Analysis, set the No. of Bins to 4 (to include only the strongest spectral components).
Send to Synthesis.
In Synthesis, the wave will look and sound like the original but without the noise.
Zeros the phases (“0 Phases” button) to make it even better.

10.7 Chirp z-Transform

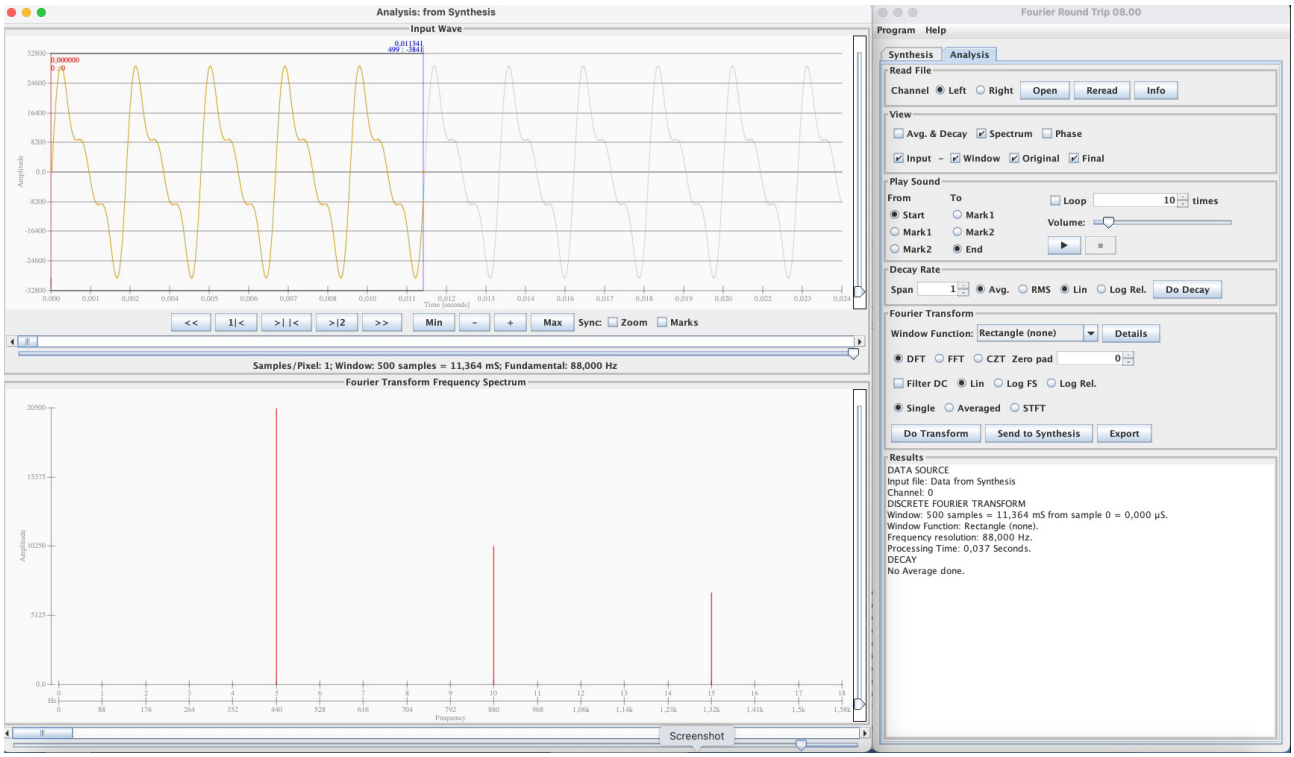


Figure 10-1 DFT of 3 Harmonics (zoomed)

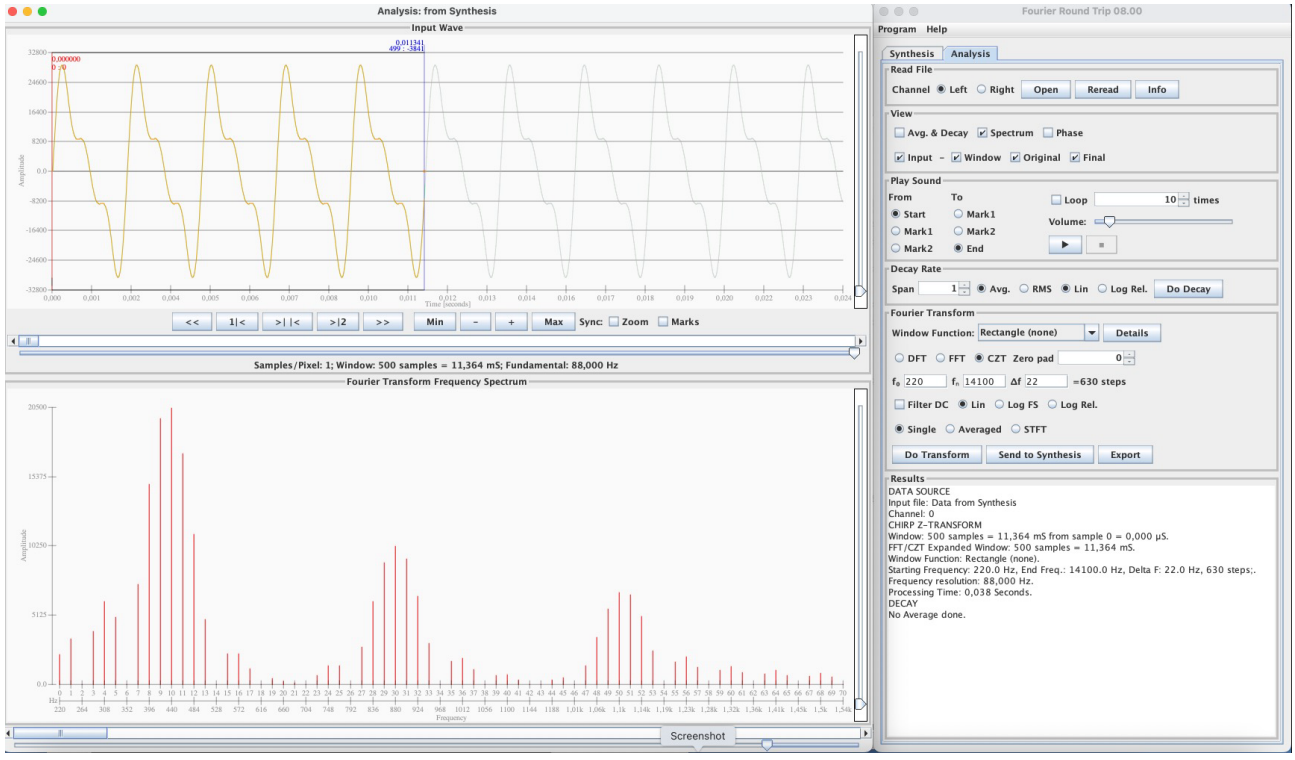


Figure 10-2 CZT of the same 3 Harmonics

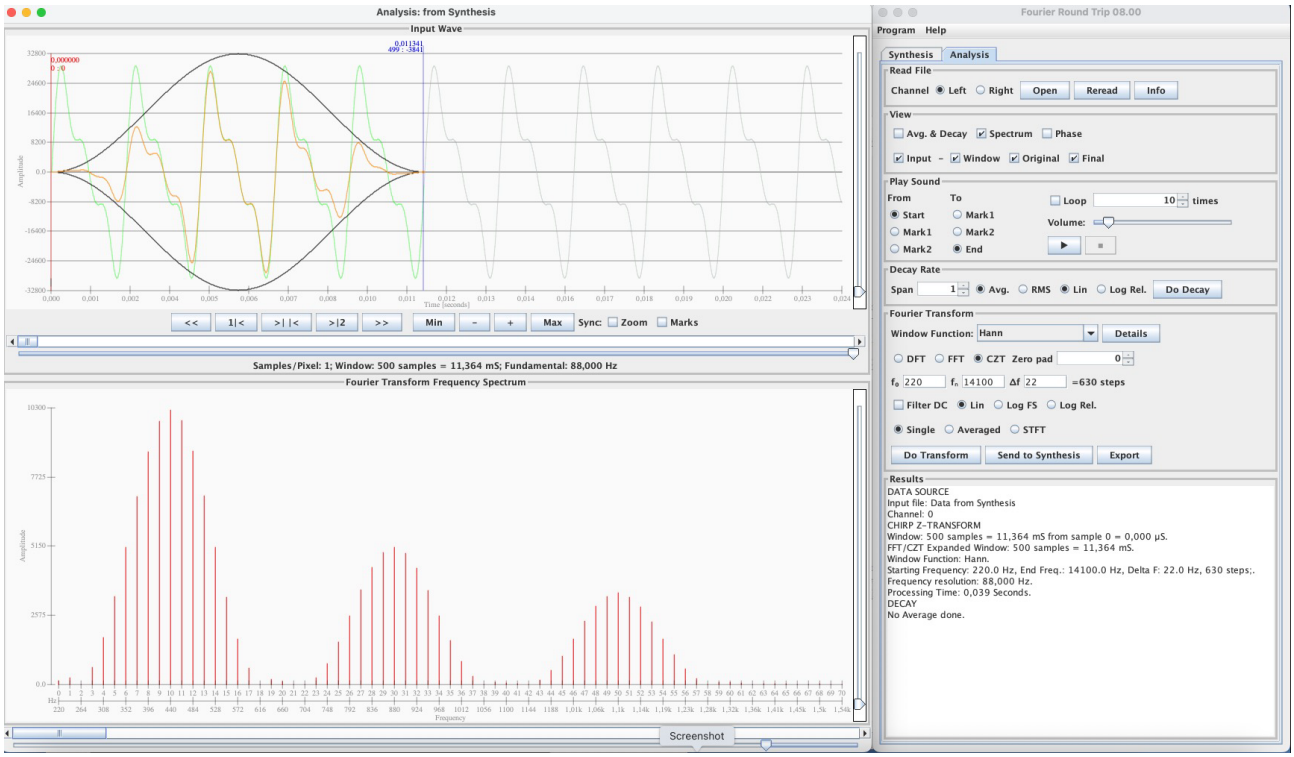


Figure 10-3 The same Harmonics, Chirp-Z Transform with Hann Window

11 Appendix Colour Export File Format

The file start with a line defining the separator followed by lines with the colour name and hexadecimal code for alpha, red, green and blue:

```
sep=,
Synthesis Background,0xfffcfcfc,The background colour of the synthesis plots.
Analysis Background,0xcffa90b,The background colour of the analysis plots.
Axis Color,0xb5000000,The colour of the axes in all plots.
Axis Graticule Color,0x3c000000,The colour of the graticule lines over the plot axes.
Plot Color,0xffff0000,The line colour for plots not otherwise defined.
Wave Color,0xffff0000,The line colour of the input wave in input and decay plots.
Reference Figure Color,0x780000ff,The line colour of the synthesis reference figure.
Constructor Figure Color,0xb4ed7d31,The line colour of the synthesis constructor figure.
Reference Figure Vector Color,0xff00cc33,The line colour of the synthesis reference figure vectors.
Reference Figure Circle Color,0x61009900,The line colour of the synthesis reference figure circles.
Reference Figure Point Color,0xb4ed7d31,The colour of points on the synthesis reference figure.
Constructor Figure Point Color,0xffff0000,The colour of the points on the synthesis constructor figure.
Sound Color,0x78a0b4a0,The colour of the synthesised wave in synthesis or the sound wave in analysis.
Sound Window Color,0x7800ff00,The colour of the sound wave within the analysis window.
Mark1 Color,0x80ff0000,The colour of Mark1.
Mark1 Label Color,0xffff0000,The colour of Mark1 labels.
Mark2 Color,0x800000ff,The colour of Mark2.
Mark2 Label Color,0xff0000ff,The colour of Mark2 labels.
Padding Color,0xb4ff8400,The colour of the zero padding at the end of the analysis window.
FFT Expansion Color,0x640000ff,The colour of the padding to expand the analysis window to a power of 2 for
an FFT.
Audio Cursor Color,0xffc88436,The colour of the audio cursor showing the position of the sound playing.
Video Cursor Color,0xffc81414,The colour of the video cursor showing the position of STFT spectrum being
shown.
Message Color,0xffff3232,The colour of messages in the analysis plots.
FT Window Color,0xb4000000,The colour of messages in the plots.
FT Overlap Color,0xb4006464,The colour of repeated window functions in analysis when averaging or STFT.
FT Final Color,0xb4ff8400,The colour of the final analysis wave after applying the window function.
FT Stale Color,0xffff0000,The colour of stale messages in analysis plots.
Average Wave Color,0xb4009600,The colour of the averaged wave in the decay plot.
Decay Interpolation Color,0xb4b400ff,The colour of the decay interpolated line.
Decay Interpolation Color Full,0xffb400ff,The colour of the decay plot labels.
Spectrum Notes Color,0x9a0000ff,The colour of the musical notes on the analysis spectrum.
Spectrum Harmonics Color,0xd6007800,The colour of the harmonics on the analysis spectrum.
```

When importing, colours whose names are not used by the current version of the program will be ignored, but listed in a message at the end of the import.
Errors in the hex codes will cause the import to stop.

12 Appendix Analysis Properties Export File Format

The file has the extension `.properties` and uses the Java Properties format.
Note: the properties are in alphabetical order.

```
#Written by FRT version 08.07
#Sun Apr 19 17:52:50 CEST 2026
CHANNEL=0
DECAY_AVG=true
DECAY_LIN=true
DECAY_PAN=5544377
DECAY_SPAN=1
DECAY_XMAG=186
DECAY_YMAG=40
FREQ_OF_A=440.0
FT_CZT_DF=1.0
FT_CZT_FN=110.0
FT_CZT_FO=100.0
FT_DFT_FFT_CZT=FFT
FT_FILTER_DC=false
FT_FRAMES=2
FT_LIN_LOGFS_LOGREL=SpectrumLinear
FT_SINGLE_AVG_STFT=FTsingle
FT_STEP=50
FT_WINDOW_FUNCTION=Rectangle (none)
FT_WINDOW_PARAMETER=1.0
FT_ZERO_PAD=0
HARMONICS_FROM=256.0
INPUT_FILE=/Users/alan/_SoundTests/Schlums/Nr2_Reise-Clavichord_Deckel-offen.wav
INPUT_PAN=5548784
INPUT_XMAG=189
INPUT_YMAG=38
MARK1_DECAY_SAMPLE=5633120
MARK1_INPUT_SAMPLE=5618296
MARK2_DECAY_SAMPLE=5665306
MARK2_INPUT_SAMPLE=5661856
MARK_SYNC=false
SHOW_DECAY=true
SHOW_FINAL=true
SHOW_FRAMES=true
SHOW_HARMONICS=true
SHOW_INPUT=true
SHOW_NOTES=true
SHOW_ORIGINAL=true
SHOW_PHASE=false
SHOW_SPECTRUM=true
SHOW_WINDOW=true
SPECTRUM_PAN=499
SPECTRUM_XMAG=1396
SPECTRUM_YMAG=11
STFT_SPEED=0.5
TEMPERAMENT=equal
TIMESTAMP=2026-04-19 17\:52\:50
ZOOM_SYNC=false
```

13 Appendix Cardioids

There are several ways of generating a cardioid, the shape of the reference and constructor figures for a wave with one overtone.

13.1 Epicycloid with Equal-sized Circles

https://en.wikipedia.org/wiki/Cardioid (Viewed 25.02.2022)

$x(\theta) = 2a(1-\cos\theta).\cos\theta$

$y(\theta) = 2a(1-\cos\theta)\sin\theta$

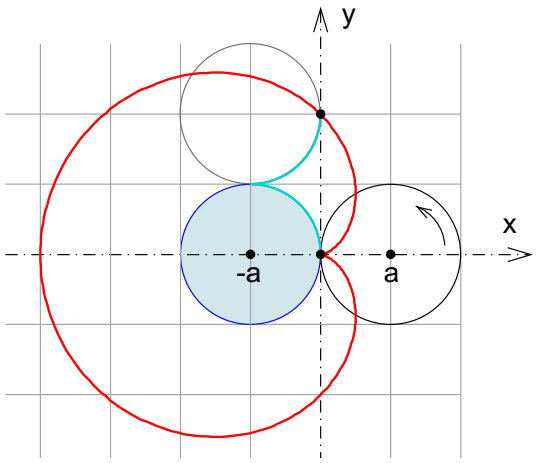


Figure 13-1 Cardioid constructed by rolling a circle round another (wikipedia)

For a=1/2 this gives

$x = \cos(\theta) - \cos^2(\theta)$

$y = \sin(\theta) - \sin(\theta)\cos(\theta)$

13.2 Epicycloid with different-sized Circles

https://en.wikipedia.org/wiki/Epicycloid

Small circle radius r going round a larger circle radius R

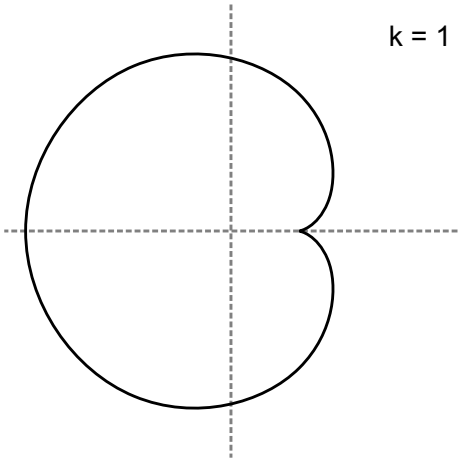
$x(\theta) = (R+r)\cos\theta - r\cos(\frac{R+r}{r}\theta)$

$x(\theta) = (R+r)\sin\theta - r\sin(\frac{R+r}{r}\theta)$

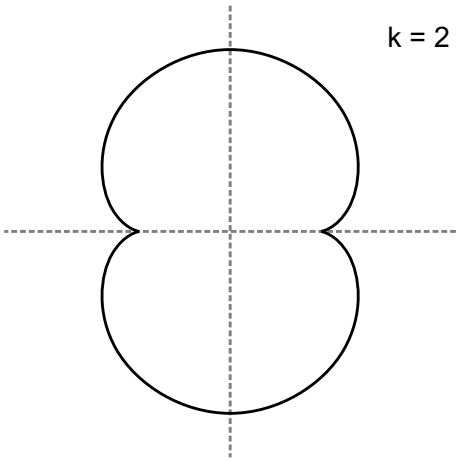
Circles of radius R = kr

$x(\theta) = r(k+1)\cos\theta - r\cos((k+1)\theta)$

$x(\theta) = r(k+1)\sin\theta - r\sin((k+1)\theta)$



k = 2 rotated 90° is 2nd overtone - nephroid



14 Appendix Proof that 2nd Harmonic Constructor is Reference Figure Translated

See Figure 3-10 – the orange constructor figure is 0.5 units to the right of the reference figure.

Reference Figure

$$y = \sin(\theta) + \frac{1}{2}\sin(2\theta)$$
$$x = \cos(\theta) + \frac{1}{2}\cos(2\theta)$$

Constructor Figure

$$y = \sin(\theta) + \frac{1}{2}\sin(2\theta)$$

$$x = \frac{y}{\tan(\theta)} = \frac{\sin(\theta) + \frac{1}{2}\sin(2\theta)}{\tan(\theta)}$$

Since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$

$$= \frac{\cos(\theta)\,[\sin(\theta) + \frac{1}{2}\sin(2\theta)]}{\sin(\theta)}$$

Since $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
and thus $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$$= \frac{\cos(\theta)\,[\sin(\theta) + \sin(\theta)\cos(\theta)]}{\sin(\theta)}$$

$$= \cos(\theta) + cos^2(\theta)$$

Since $cos^2(x) = \frac{1}{2}(1 + \cos(2x)$

$$= \cos(\theta) + \frac{1}{2}[1 + \cos(2\theta)]$$

$$= \cos(\theta) + \frac{1}{2}\cos(2\theta) + \frac{1}{2}$$

This is the same pair of equations as the reference figure, but with x translated by $+\frac{1}{2}$ as we see it in Figure 3-10 lower diagram.

Note also that the original reference figure equations are easily transformed into the equations of the cardioid (section 13.1):

$$y = \sin(\theta) + \frac{1}{2}\sin(2\theta)$$
$$= \sin(\theta) + \sin(\theta)\cos(\theta)$$

And as shown above

$$x = \cos(\theta) + cos^2(\theta)$$

(The signs are inverted as we have the cardioid reversed.)

15 Appendix Exponential Decay

Exponential decay is defined by the formula:

$$y(t) = Ae^{-t/\tau}$$

where y(t) is the resulting amplitude at time t, A is the initial amplitude, τ is the time constant which is the time it takes for y(t) to decay by 1/*e*.
In acoustics, e.g. of concert halls, the standard for specifying decay time is the time it takes to decay by -60dB, denoted by t_{60}:

$$20log_{10}\left(\frac{y(t_{60})}{y(0)}\right) = -60dB$$

or

$$\frac{y(t_{60})}{y(0)} = 10^{-60/20} = 0.001$$

thus

$$t_{60} = -ln(0.001)\tau \approx 6.91\tau$$

and

$$\tau = -\frac{t_{60}}{ln(0.001)}$$

The program uses the entered value for t_{60} to calculate the time constant and apply the corresponding exponential decay.

16 Appendix Interpolation and Peak Detection

16.1 Overview

The simplest form of interpolation is to use zero padding on the input wave. Doubling the number of samples in the input window with zero padding doubles the number of data points in the resulting DFT. Note that this does not increase the resolution. [Smith] p. 26ff.
[Smith] p. 186ff. also advocates quadratic interpolation (Quadratically Interpolated Fast Fourier Transform QIFFT as an approximate maximum likelihood method for spectral peak estimation).
We do not treat phase.
Since the various methods use different symbols for the same terms, we attempt to unify them.

16.2 Quadratic Interpolation

Taking the peak (middle) sample and the left and right neighbours with amplitudes denoted by l, m, r respectively
The interpolated bin *b'* is

$$b' = \frac{1}{2}\frac{l-r}{l-2m+r}$$

and the interpolated peak amplitude *y* is

$$y = m - \frac{1}{4}(l-r)b$$

This is unbiased (exact) for a Gaussian window in dB or if the window contains an exact number of cycles. Also if l=r, m is the exact peak.

16.3 Jain

The original [Jain] uses 2 data points.

$$a = \frac{|r|}{|m|}$$

$$d = \frac{a}{(1+a)}$$

$$b' = b + d$$

and the interpolated amplitude *y* is
if r>m

$$y = \frac{2\pi(1-d)}{N}\frac{|r|}{|sin(\pi(1-d))|}$$

else

$$y = \frac{2\pi d}{N}\frac{|m|}{|sin(\pi(1-d))|}$$

Note: the factor 2/N applies to an unscaled DFT. Our transform scales by 2/N to give amplitudes normalised to 1, so the factor is not applied again during the interpolation[7].

16.4 Grandke

[Grandke] gives an algorithm specifically for the Hann window using two frequency bins *m* and *r*.

[7] Thanks to Claude Sonnet 4.6 for pointing this out.

$$\alpha = \frac{r}{m}$$

$$d = \frac{2\alpha - 1}{\alpha + 1} = \frac{2r - m}{r + m}$$

$$b' = b + d$$

The amplitude y is

$$y = \frac{2\pi d(1-d)}{sin(\pi d)} e^{-\pi i d}(1+d)m$$

$$y = \frac{2\pi d(1-d)}{sin(\pi d)} e^{-\pi i d}(d-2)r$$

using the equation for the larger line, *m* or *r*.
The program will give a warning if the selected window is not Hann, but does not prevent the user from running the interpolation.
Note: the factor 2 in the amplitude is not applied because the transform already applies it to give normalised amplitudes (cf. Jain above).

16.5 Other Algorithms

The following are variants of the above which have not been implemented, included here for completeness.

16.5.1 Variants of Jain

[Minda] and [Cabadag] give a version of Jain using 3 data points:
If l>r

$$d = \frac{l}{l+m} - 1$$

else

$$d = \frac{r}{m+r}$$

and

$$b' = b + d$$

and the interpolated amplitude *y* is not given.

[Minda] does an FFT of the original signal and an FFT of the signal with a padding of 2 zeros and uses the above methods depending on whether the middle sample of the latter is less than the middle sample of the former. This solves a problem that can occur if *l* and *m* are on the main lobe, but *r* is on a side lobe and *r*>*l*.

17 Appendix Inharmonicity

Ideal strings, when struck or plucked, resonate with harmonics that are exact multiples of the fundamental frequency. Real strings have stiffness which causes the harmonics to increase in frequency according to the formula (from [Fletcher]):

$$f_n = nf_0\sqrt{(1 + n^2B)}$$

where f_0 is the fundamental frequency, f_n is the frequency of the n^{th} harmonic and B is the coefficient of inharmonicity.

B can be calculated from Young's modulus, the diameter, length and tension of the string, but as these are unknown, we obtain B by rearranging the above formula:

$$B = \frac{\left(\frac{f_n}{nf_0}\right)^2 - 1}{n^2}$$

The result is shown multiplied by 10^6.

The inharmonicity is also shown in cents as:

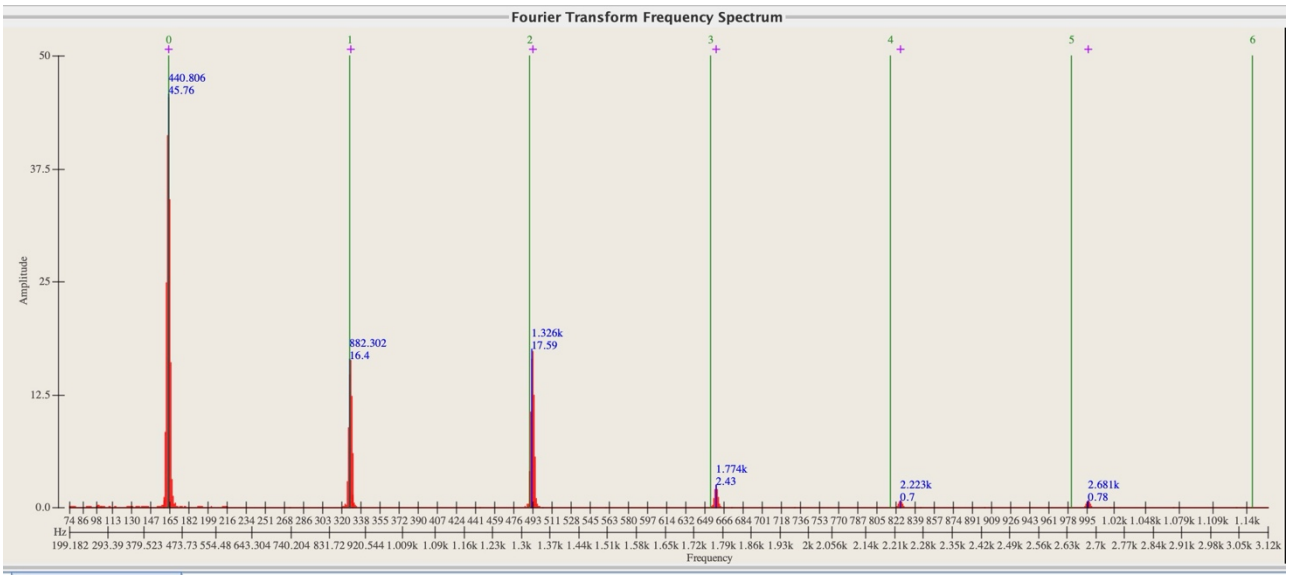
$$cents = 1200log_2\left(\frac{f_n}{nf_0}\right)$$

If the harmonic's frequency is lower than calculated, B and the cents are negative.

Note that the f_0 is taken to be the interpolated frequency of the first peak which may not correspond to the nominal frequency of the note.

To obtain useful values from the program, care is needed in selecting the peaks. They must correspond to the fundamental and harmonics of the spectrum, which can easily be found by selecting the harmonics in the view section and setting the "from" field to the interpolated fundamental frequency. If the desired peaks cannot be found automatically by adjusting the start bin, skip and threshold, they can be adjusted manually in the peak bins field. The program uses the interpolated frequencies for f_0 and f_n to calculate B.

An example is shown in Figure 17-2. The start bin was set to 150 to ignore the low frequency noise and a threshold of 1.0 was set to ignore intermediate non-harmonic frequencies. However, this also included a spurious peak before the fundamental and excluded the weak third and fourth harmonics, so the bin numbers of their peaks were adjusted manually in the peak bins field – the bin numbers were found by zooming in to the spectrum and reading them off the x-axis.



Interpolation

Start Bin 150 No. Peaks 7 6 Skip 0 Threshold 1.0

Find Peak Bins frame 0 164 328 493 659 826 996

Method Quadratic Interpolate

Peaks and Interpolations

No. of Peaks	7
Start Bin	150
Bins Skipped	0
Threshold	1.0
Interpolation Method	Quadratic

Bin	Freq.	Amplitude	Inharmonicity*10^6	Cents
164	440.806 Hz	45.76	-	-
328	882.302 Hz	16.4	391	1.353
493	1.326k Hz	17.59	609	4.732
659	1.774k Hz	2.43	759	10.452
826	2.223k Hz	0.7	707	15.157
996	2.681k Hz	0.78	767	23.580

Figure 17-2 Example of Inharmonicities of Note A4

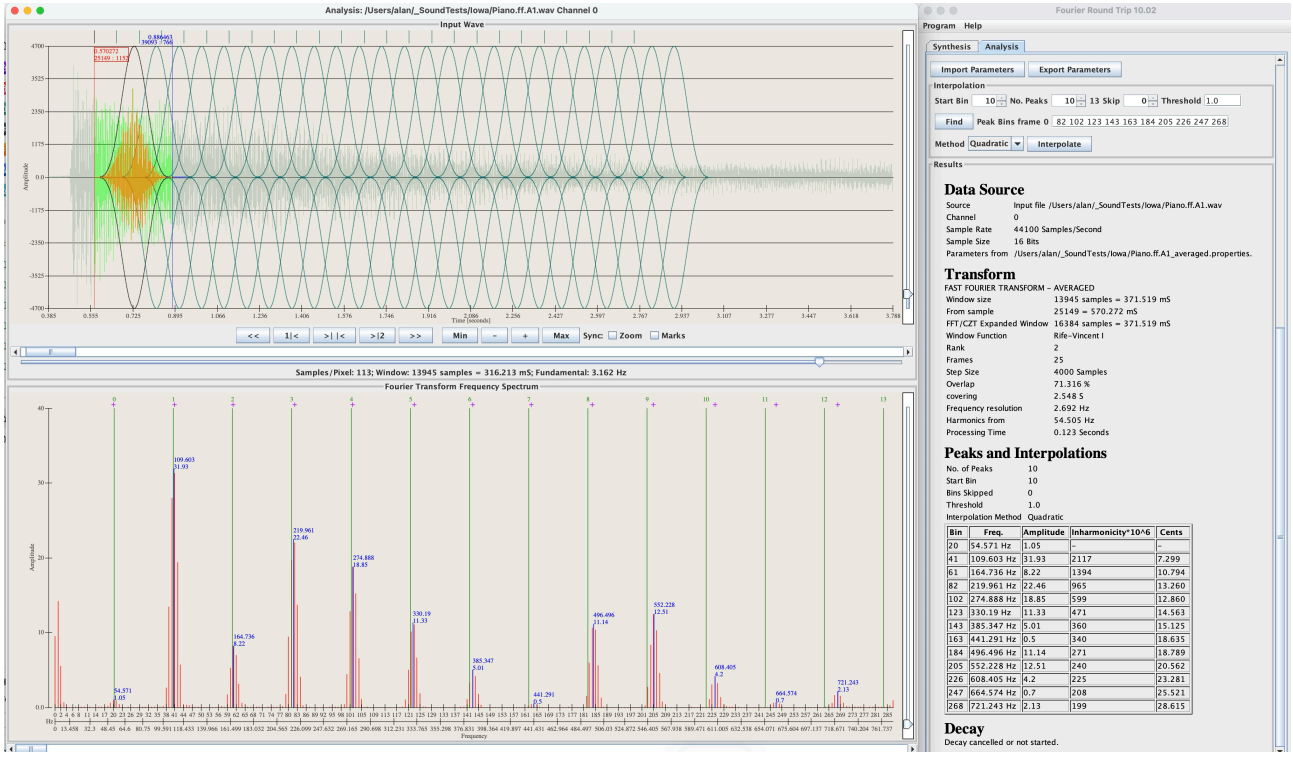


Figure 17-1 Example of Inharmonicities of Note A1

18 Appendix Window-Functions

A selection of window-functions is provided. Their basic characteristics and coefficients are summarised here. For further information see the literature below.

18.1 TBD Add

18.1.1 Force

https://www.uml.edu/docs/Windows-Leakage_tcm18-191248.pdf
i.e. Force-Exponential
https://pearl-hifi.com/06_Lit_Archive/15_Mfrs_Publications/10_Bruel_Kjaer/05_Technical_Reviews/Proper_Use_of_Weighting_Functions_for_Impact_Testing.pdf

18.1.2 Parzen

Defining $L \triangleq N+1$, the Parzen window, also known as the **de la Vallée Poussin window**,[14] is the 4th order *B*-spline window given by:

$$w_0(n) \triangleq \begin{cases} 1 - 6\left(\frac{n}{L/2}\right)^2 \left(1 - \frac{|n|}{L/2}\right), & 0 \le |n| \le \frac{L}{4} \\ 2\left(1 - \frac{|n|}{L/2}\right)^3 & \frac{L}{4} < |n| \le \frac{L}{2} \end{cases}$$

$$w[n] = w_0\left(n - \frac{N}{2}\right),\ 0 \le n \le N$$

https://en.wikipedia.org/wiki/Window_function#Parzen_window

18.1.3 Poisson

See Harris 1978

18.1.4 Cauchy

See Harris 1978

18.1.5 Others

See Evaluation_of_Various_Window_Functions_using_Multi-Instrument_D1003.pdf
See Window function - Wikipedia.pdf and cf. Wikipedia original.

18.1.6 Generalised Adaptive Polynomial

See Justo and Beccaro paper 2020.
Also Sun, Liu, Cai & Long “A Novel Method for Designing General Window Functions with Flexible Spectral Characteristics”.

18.2 Rectangle (none)

```
windowFunction[i] = 1.0;
```

The wave is unchanged.

18.3 Bartlett

As defined in [Heinzel, who also references [Harris:

```
double z = 2.0 * i / windowWidth;
if (z <= 1.0) {
      windowFunction[i] = z;
} else {
      windowFunction[i] = 2.0 - z;
}
```

This is a type of triangular window.

18.4 Welch

From [Heinzel

```
double z = 2.0 * i / windowWidth;
windowFunction[i] = 1 - Math.pow((z - 1), 2);
```

This is equivalent to https://en.wikipedia.org/wiki/Window_function#Welch_window which gives:

> The Welch window consists of a single parabolic section:
>
> $$w[n] = 1 - \left(\frac{n - \frac{N}{2}}{\frac{N}{2}} \right)^2, \quad 0 \le n \le N.$$[20]
>
> The defining quadratic polynomial reaches a value of zero at the samples just outside the span of the window.

18.5 Hamming

As defined in [Heinzel and [Harris:

```
windowFunction[i] = 0.54 - (0.46 * (Math.cos(Math.PI*2*i/windowWidth)));
```

18.6 Hann

```
windowFunction[i] = 0.5 * (1-(Math.cos(Math.PI*2*i/windowWidth)));
```

Taken from [Wikipedia and [Heinzel rather than [Harris as the latter gives the function with the sign reversed.

18.7 Cosine Sum Windows

18.7.1 Formula

These are calculated by a sum of cosine terms with varying coefficients:

```
windowFunction[i] = a0 – a1cos(2πi/N) + a2cos(4πi/N) – a3cos(6πi/N) …
```

where N is the window width and a_0, a_1, a_2, etc. are coefficients which define the window.
Blackman from https://en.wikipedia.org/wiki/Window_function#Blackman_window
Exact Blackman from https://en.wikipedia.org/wiki/Window_function#Blackman_window
Blackman-Harris-4 from https://www.mathworks.com/help/signal/ref/blackmanharris.html
Blackman-Harris-7 from https://dsp.stackexchange.com/questions/51095/seven-term-blackman-harris-window

18.7.2 Blackman and Blackman-Harris Windows

Coefficient	Blackman	Exact Blackman	Blackman-Harris-4	Blackman-Harris-7
a_0	0.42	7938/18608	0.35875	0.27105140069342
a_1	-0.5	-9240/18608	-0.48829	-0.43329793923448
a_2	0.08	1430/18608	0.14128	0.21812299954311
a_3			-0.01168	-0.06592544638803
a_4				0.01081174209837
a_5				-0.00077658482522
a_6				0.00001388721735

18.7.3 Nuttall Windows

From

[Nuttall]	Albert H. Nuttall. Some Windows with Very good Sidelobe Behavior. IEEE Transactions on Acoustics, Speech and Signal Processing, Vol. ASSP-29, No.1 February 1981. https://ieeexplore.ieee.org/document/1163506

and [Heinzel

Coefficient	Nuttal-3	Nuttal-3a	Nuttall-3b	Nuttall-4	Nuttall-4a	Nuttall-4c
a_0	0.375	0.40897	0.4243801	0.3125	0.338946	0.355768
a_1	-0.5	-0.5	-0.4973406	-0.46875	-0.481973	-0.487396
a_2	0.125	0.09103	0.0782793	0.1875	0.161054	0.144232
a_3				-0.03125	-0.018027	-0.012604

18.8 Windows with Parameters

18.8.1 Gaussian

From [Wikipedia (which differs from [Harris)

```
windowFunction[i] = Math.exp(-0.5 * Math.pow((i-m) / (gaussianSigma*m),2));
```

$$w[n] = e^{-\frac{1}{2}\left(\frac{n-N/2}{\sigma N/2}\right)^2}, 0 \leq n \leq N$$

The variable σ GaussianSigma is entered by the user and should be <0.5.
[Harris:

$$w[n] = e^{-\frac{1}{2}\left(\alpha\frac{n}{N/2}\right)^2}, -N/2 \leq n \leq N/2$$

Where $\alpha = 1/\sigma$

18.8.2 Dolph-Chebyshev

This is programmed according to [Lyons-DC].
See also:
https://en.wikipedia.org/wiki/Window_function#Dolph–Chebyshev_window
The Sidelobe value for the parameter is the decibels of the sidelobe below the main lobe – the alternative parameter alpha is this value / 20, e.g. for 60dB alpha is 3.

18.8.3 Kaiser-Bessel

From [Heinzel, [Harris, [Wikipedia.
The variable α is entered by the user and [Heinzel gives characteristics for values between 2.0 and 7.0, and [Wikipedia gives 3.0 as a typical value.
We find two forms of the formula:
[Heinzel and [Wikipedia:

$$w[i] = \frac{I_0\left(\pi\alpha\sqrt{1-\left(\frac{2i}{N}-1\right)^2}\right)}{I_0(\pi\alpha)}$$ [Heinzel for i = 0…N-1; [Wikipedia 0<=i<=N EQ1

[Harris:

$$w[i] = \frac{I_0\left(\pi\alpha\sqrt{1-\left(\frac{2i}{N}\right)^2}\right)}{I_0(\pi\alpha)}$$ For 0<=|i|<=N/2 EQ2

Where I_0 is the zero-order Bessel function of the first kind:

$$I_0(x) = \sum_{k=0}^{\infty}\left[\frac{\left(x/2\right)^k}{k!}\right]^2$$

We use EQ1 with i = 0 …N-1.

18.8.4 Tukey

From [Wikipedia, which gives these errors in [Harris:

> *Harris 1978 (p 67, eq 38) appears to have two errors: (1) The subtraction operator in the numerator of the cosine function should be addition. (2) The denominator contains a spurious factor of 2. Also, Fig 30 corresponds to α=0.25 using the Wikipedia formula, but to 0.75 using the Harris formula. Fig 32 is similarly mislabeled.*

$$\left.\begin{aligned} w[n] &= \frac{1}{2}\left[1-\cos\left(\frac{2\pi n}{\alpha N}\right)\right], \quad 0 \le n < \frac{\alpha N}{2} \\ w[n] &= 1, \frac{\alpha N}{2} \le n \le \frac{N}{2} \\ w[N-n] &= w[n], 0 \le n \le \frac{N}{2} \end{aligned}\right\}$$

For α=0 it becomes rectangular and for α=1 it becomes a Hann window.

18.8.5 Exponential or Poisson

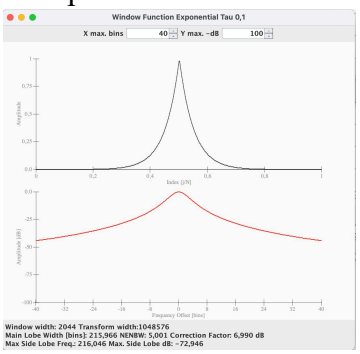
For use with transients longer than the record length (ringing).

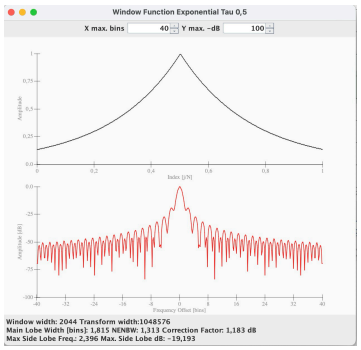
$$w(n) = e^{-|n-\frac{N}{2}|\frac{1}{\tau}}$$

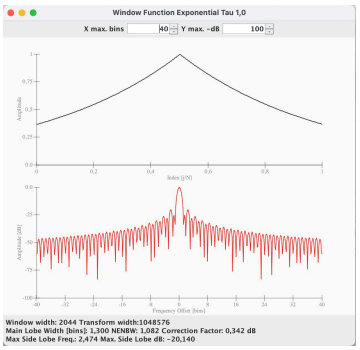
τ is the time constant

https://en.wikipedia.org/wiki/Window_function#Exponential_or_Poisson_window
Brüel&Kjaer TR3,1987

Example values of τ:







See also [Smith]: https://www.dsprelated.com/freebooks/sasp/Spectrum_Analysis_W indows.html

18.9 Rife-Vincent Windows

These were first presented in in [Rife-Vincent], but their use does not appear to be very widespread. Each class has multiple variants which we shall call "ranks". The rank is set as a parameter.

18.9.1 Class 1

Rife-Vincent Class I is designed to give the minimum high-order side lobe amplitude and fastest decay of the side lobes but have a wide main lobe.
[Rife-Vincent] gives the following equation for class I:

$$h(t) = h_\tau(t)\left[1 + \sum_{n=1}^{M} D_n cos\left(\frac{2\pi nt}{T}\right)\right]$$

The coefficient D_n for each rank M are calculated as described in the paper up to M=10. M=0 is omitted as it is the same as the rectangular window.
The paper gives values up to M=4 as follows:

	n=0	n=1	n=2	n=3	n=4
M=0	1				
M=1	1	1			
M=2	1	4/3	1/3		
M=3	1	3/2	3/5	1/10	
M=4	1	8/5	4/5	8/35	1/35

and the calculated values agree with these. [Duda] gives values up to M=6.
Some sources include the sign in the coefficients, others implement the alternating sign with $(-1)^m$.

The factor $h_\tau(t)$ is the rectangular window to set the result to 0 outside the limits $0 \leq n < N$.

M=0 is the rectangular window.
M=1 is the same as Hann.
M=2 is the same as Nuttall-3
M=3 is the same as Nuttall-4
M=4 is the same as Flat Top Maximum Sidelobe Decay (not included here yet).

18.9.2 Class II

Rife-Vincent Class II windows have minimum main-lobe width.
The coefficients are more difficult to calculate than for class I and the details are not given here.
They have an additional parameter R, a measure of the side-lobe suppression in dB, similar to Dolph-Chebyshev (see section 18.8.2).

To obtain the desired side-lobe suppression, [Rife-Vincent] suggest minimum values for the rank M as follows:

Side-lobe suppression dB	Minimum M
36	3
42	4
48	5
54	6
60	7
66	9

The program does not enforce these.

18.9.3 Class III

Rife-Vincent Class III windows are a compromise between class I and class II: "*Class III weighting provides better resolution than Class I for tones with 'small' frequency separation. Moreover, they also provide better resolution than Class II of tones with a 'large' frequency separation."*
The equation given was not practical to implement, so only ranks 2-4 are included using Table III in [Rife-Vincent].

18.10 Flat Top Fast Decaying Windows

From [Heinzel

Coefficient	Flat-Top-Fast-3	Flat-Top-Fast-4	Flat-Top-Fast-5
a_0	0.26526	0.21706	0.1881
a_1	-0.5	-0.42103	-0.36923
a_2	0.23474	0.28294	0.28702
a_3		-0.07897	-0.13077
a_4			0.02488

18.11 Flat Top Minimum Side-Lobe Windows

From [Heinzel

Coefficient	Flat-Top-Min.Lobe-3	Flat-Top-Min.Lobe-4	Flat-Top-Min.Lobe-5
a_0	0.28235	0.241906	0.209671
a_1	-0.52105	-0.460841	-0.407331
a_2	0.19659	0.255381	0.281225
a_3		-0.041872	-0.092669
a_4			0.0091036

18.12 Windows from Commercial Spectrum Analysers

From [Heinzel

Coefficient	National Instruments Flat-Top	Hewlett Packard	Stanford Research SR785
a_0	0.2810639	1.0	1.0
a_1	-0.5208972	-1.912510941	-1.93
a_2	0.1980399	1.079173272	1.29
a_3		-0.1832630879	-0.388
a_4			0.028

18.13 Flat-Top Windows Developed for GEO600 Gravitational Waves

From [Heinzel

Coefficient	HFT70	HFT95	HFT90D	HFT116D	HFT144D
a_0	1.0	1.0	1.0	1.0	1.0
a_1	-1.90796	-1.9383379	-1.942604	-1.9575375	-1.96760033
a_2	1.07349	1.3045202	1.340318	1.4780705	1.57983607
a_3	-0.18199	-0.4028270	-0.440811	-0.6367431	-0.81123644
a_4		0.0350665	0.043097	0.228389	0.22583558
a_5				-0.0066288	-0.02773848
a_6					0.00090360

Coefficient	HFT169D	HFT196D	HFT223D	HFT248D
a_0	1.0	1.0	1.0	1.0
a_1	-1.97441842	-1.979280420	-1.98298997309	-1.985844164102
a_2	1.65409888	1.710288951	1.75556083063	1.791176438506
a_3	-0.95788186	-1.081629853	-1.19037717712	-1.282075284005
a_4	0.33673420	0.448734314	0.56155440797	0.667777530266
a_5	-0.06364621	-0.112376628	-0.17296769663	-0.240160796576
a_6	0.00521942	0.015122992	0.03233247087	0.056656381764
a_7	-0.00010599	-0.000871252	-0.00324954578	−0.008134974479
a_8		0.000011896	0.00013801040	0.000624544650
a_9			-0.00000132725	-0.000019808998
a_{10}				0.000000132974

19 Appendix Chirp z-Transform

Note: this has not been implemented as modern processors are fast enough not to need the FFT convolution method. The program implements the transform directly.

$$X[k] = \sum_{k=0}^{M} \sum_{n=0}^{N} x[n] e^{-2\pi(\delta t * n)(f_{0+(\delta f * k)})}$$

Where:
x[n] = input sample n
N = number of samples in input
M = number of resulting frequencies
δt = time between input samples
f_0 = first frequency
δf = desired frequency spacing
X[k] = frequency sample k

The following shows my understanding of the chirp z-transform based on [Oppenheim (p. 656-661) and [Rabiner for future reference.
The steps are described in [Rabiner, but as these are for the general case of a spiral in the z-plane and we wish to remain on the unit circle, we use $A_0 = 1$ and $W_0 = 1$, as in [Oppenheim.
Note the following equivalences in notation:

[Oppenheim	**[Rabiner**
ω	$2\pi\Theta$
ω_0	$2\pi\Theta_0$
$\Delta\omega$	$2\pi\phi_0$
x[n]	x_n
g[n]	y_n
$X(e^{j\omega n})$	X_k
$e^{j\omega_0}$	$A = A_0 e^{j2\pi\theta_0}$ (A_0=1)
$W = e^{-j\Delta\omega}$	$W = W_0 e^{2\pi\phi_0}$ (W_0=1)

Basics:

$$\omega = 2\pi f$$
$$e^{j\theta} = cos\theta + jsin\theta$$
$$a^b * a^c = a^{(b+c)}$$
$$(a^b)^c = a^{(b*c)}$$
$$(a/b)^c = a^c/b^c$$

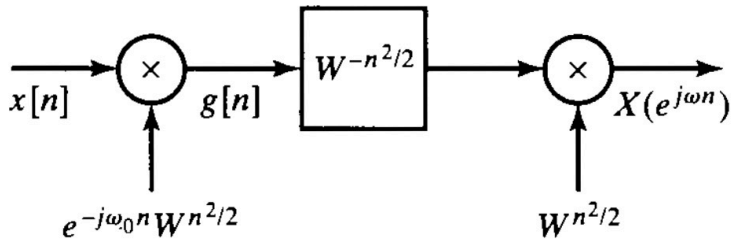


Figure 19-2 Oppenheim Fig. 9.26

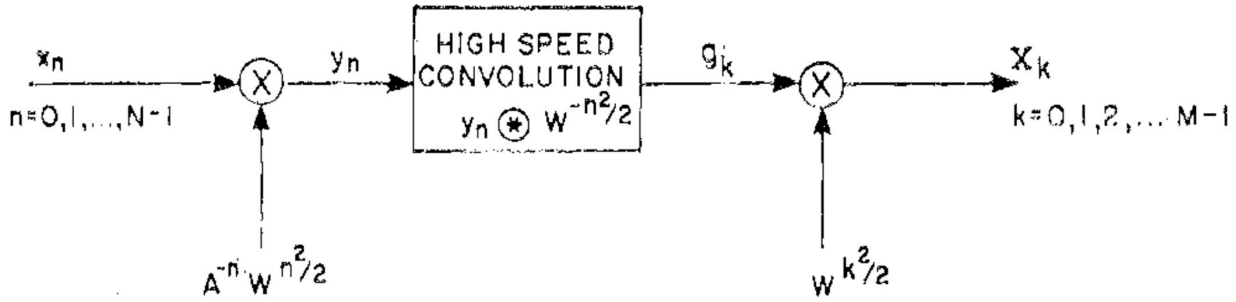


Figure 19-2Rabiner Fig. 3

For the algorithm we use frequency and convert ω = 2πf.
The chirp for the input "a" is:

$$e^{-j\omega_0} W^{\frac{n^2}{2}} = e^{-j2\pi f_0} \left(e^{-j2\pi\Delta f}\right)^{\frac{n^2}{2}} = e^{-j2\pi f_0} e^{-j2\pi\Delta f n^2/2} = e^{-j2\pi\left(f_0+\Delta f n^2/2\right)}$$

for 0<=n<=N-1, where N is the number of input samples.

The chirp for the output "b" is:

$$W^{\frac{k^2}{2}=}\left(e^{-j2\pi\Delta f}\right)^{\frac{k^2}{2}} = e^{-j2\pi\Delta f k^2/2}$$

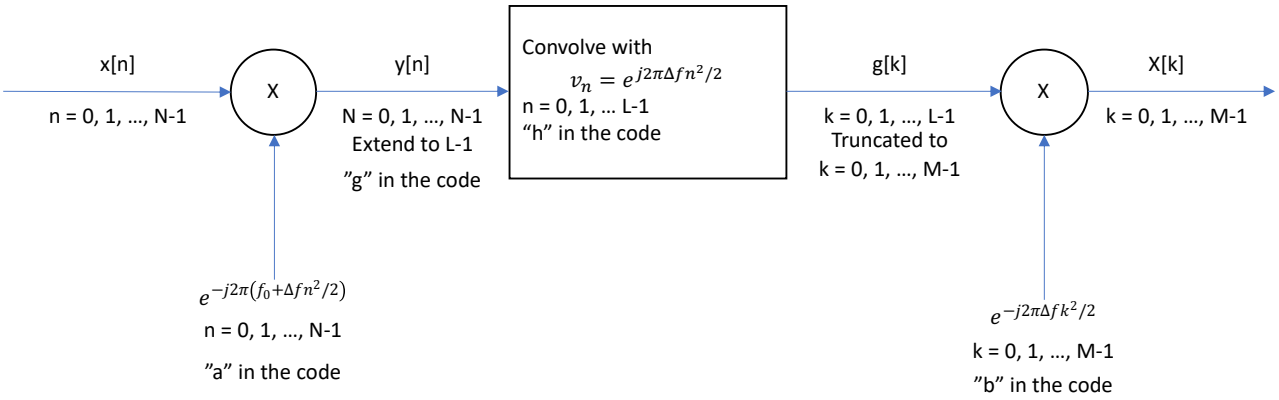
where 0<=k<=M-1, where M is the number of output frequencies.

The chirp for the convolution "h" is:

$$W^{\frac{-n^2}{2}=}\left(e^{-j2\pi\Delta f}\right)^{\frac{-n^2}{2}} = e^{j2\pi\Delta f n^2/2}$$

Where 0<=n<=L, where L is the next power of 2 greater than M+N+1 for the FFT.

To summarise:



20 Literature

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